**Al-Hamdaniya University**

**College of Education**

**Physics Department**

**Third class**

# Analytical mechanics

# محاضرات الميكانيك التحليلي لطلبة المرحلة الثالثة

## Survey to subject of analytical mechanics for third year of physics

**CHAPTER (1):** Fundamental Concepts Vectors .

**CHAPTER (2):** Newtonian Mechanics.

**CHAPTER (3):** General Motion of a Particle in Three Dimensions General .

**CHAPTER (4):** Gravitation and Central Forces Gravitational Force between a Uniform Sphere and a Particle.

**CHAPTER (5):** Dynamics of Systems of Particles Center of Mass and Linear Momentum of a System.

## References:

1- Golwala, Sunil. "Lecture notes on classical mechanics for physics 106ab." Publisher: CreateSpace Independent Publishing Platform (2014).

2-

كتاب الميكانيك التحليلي
المؤلف / كرانك ر . فاولس
ترجمة الدكتور / طالب ناهي الخفاجي

## 1.1 Vector Definition

The motion of dynamical systems is typically described in terms of two basic quantities: ***Scalars and Vectors.***

حركة الانظمة الديناميكية توصف عادة بدلالة كميات عددية او اتجاهية

A **scalar** is a physical quantity that has magnitude only.

It is completely specified by a *single number*, in *appropriate units*. Its value is *independent* of any *coordinates* chosen to describe the motion of the system.

**Examples** of scalars include mass, density, volume, temperature, and energy.

**Mathematically**, scalars are treated as real numbers. They obey all the normal algebraic rules of addition, subtraction, multiplication, division, and so on.

***Scalar represented* as**

A, B, C… without upper notation or letter with brackets: $|\vec{A}|, |\vec{B}|, |\vec{C}|$….

الكمية العددية هي كمية مادية لها مقدارفقط.

يتم تعريف الكمية العددية بشكل كافي برقم واحد، بالوحدات المناسبة. وتكون قيمتها مستقلة عن أي إحداثيات مختارة لوصف حركة النظام.

من الأمثلة على الكميات العددية الكتلة والكثافة والحجم ودرجة الحرارة والطاقة.

رياضيا، تعامل الكمية العددية كأرقام حقيقية. تخضع لجميع القواعد الجبرية الطبيعية للجمع والطرح والضرب والقسمة، إلخ.

A **Vector** has both magnitude and direction.

Unlike a scalar, a vector requires a *set of numbers* for its complete specification. The values of those numbers are coordinate system *dependent*.

**Examples** of vectors include displacement in space; other examples of vectors include velocity, acceleration, and force.

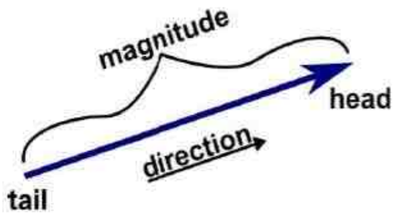
**Mathematically**, vectors combine with each other according to the parallelogram rule of addition.

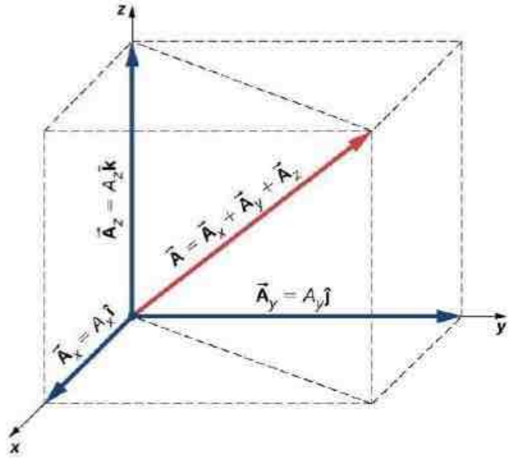
**V*ector represented* as** $\vec{A}, \vec{B}, \vec{C}$ with upper notation.

الكمية الاتجاهية كمية تتحدد بكل من المقدار والاتجاه.

على عكس الكمية العددية ، تتطلب الكمية الاتجاهية لوصفها بشكل كامل مجموعة أرقام تعتمد على احداثيات النظام.

من الأمثلة على الكميات المتجه الازاحة و السرعة والتعجيل والقوة.




In equation form **Vector** given as:

$\vec{A} = [A_x, A_y, A_z]$

$A_x, A_y, A_z$ projections of $\vec{A}$ along the coordinate axes $x, y, z$.

## 1.2 Vector Algebra

- **Equality of Vectors**

Two vectors are equal if, and only if, their respective components are equal.

يتساوى المتجهان اذا تساوت مركباتهم المتناظرة والمتجهين المتساويين يكونان متوازين ولهما نفس الطول. لكن ليس بالضروري لهما نفس الموضع.

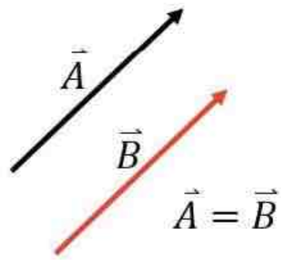
$\vec{A} = \vec{B}$

If and only if

$A_x = B_x, \quad A_y = B_y, \quad A_z = B_z$

Or

$[A_x, A_y, A_z] = [B_x, B_y, B_z]$




- **Negative Vectors**

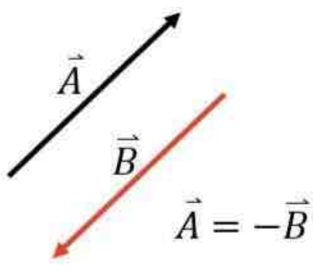
سالب اي متجه يكون متجه جديد معاكس بالاتجاه كل مركبة فيه تساوي سالب المركبة المناظرة في المتجه الاول

$\vec{A} = -\vec{B}$

If and only if

$A_x = -B_x, \quad A_y = -B_y, \quad A_z = -B_z$

Or $[A_x, A_y, A_z] = -[B_x, B_y, B_z]$




- **Vectors Addition and Subtraction**

جمع اي متجهين يكون متجه جديد مركباته توجد من خلال جمع المركبات المتناظرة في المتجهين

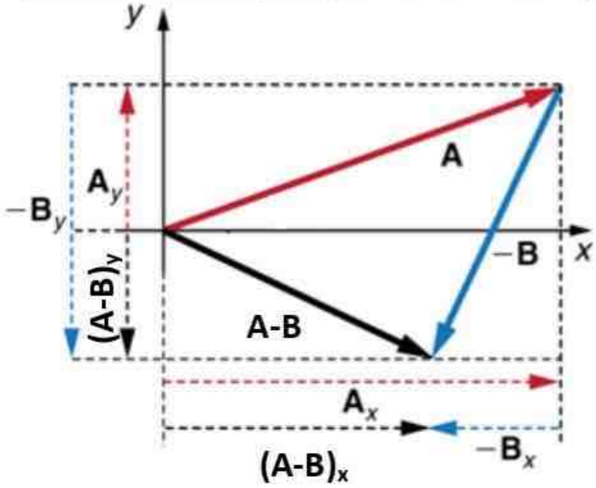
The **addition** of two vectors is defined by the equation

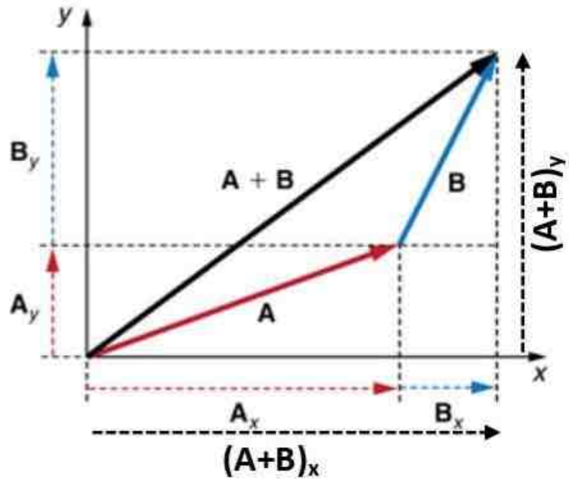
$\vec{A} + \vec{B} = [A_x, A_y, A_z] + [B_x, B_y, B_z]$

$\vec{A} + \vec{B} = [A_x + B_x, A_y + B_y, A_z + B_z]$

The **subtraction** of two vectors is defined by the equation

$$\vec{A}-\vec{B}=\vec{A}+\left(-\vec{B}\right) = [A_x - B_x\,, A_y - B_y\,, A_z - B_z]$$



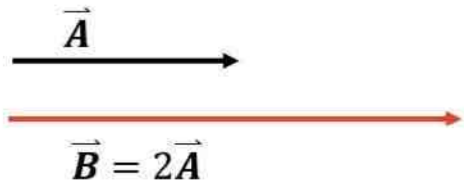

- **Multiplication by a Scalar**

عند ضرب المتجه بمقدار ثابت يتكون متجه جديد يتغير طوله اعتمادا على مقدار الثابت

If c is a scalar and A is a vector,

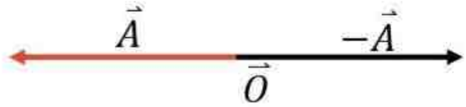
$C\vec{A} = C[A_x,\, A_y,\, A_z]$

$C\vec{A} = [CA_x,\, CA_y,\, CA_z] = C\vec{A}$




- **The Null Vector**

The vector $\vec{O} = [0, 0, 0]$ is called the **null** vector. The direction of the null vector is undefined. $\vec{A} - \vec{A} = \vec{O} = 0$




المتجه الصفري هو المتجه الذي جميع مركباته تساوي صفر

- **The Commutative Law of Addition** جمع المتجهات يخضع لقانون التبادل

$\vec{A} + \vec{B} = \vec{B} + \vec{A}$ **(H.W) Prove that**

- **The Associative Law** جمع المتجهات يخضع لقانون التجميع (ترتيب الحدود)

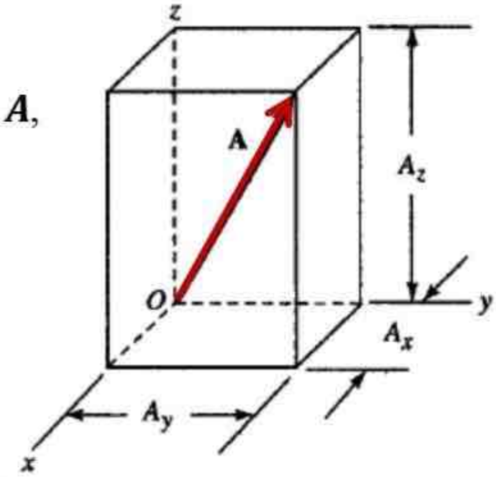
$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$ **(H.W) Prove that**

- **The Distributive Law** جمع المتجهات يخضع لقانون التوزيع

$C(\vec{A} + \vec{B}) = C\vec{A} + C\vec{B}$ **(H.W) Prove that**

$C$ is scalar.

- **Magnitude of a Vector**



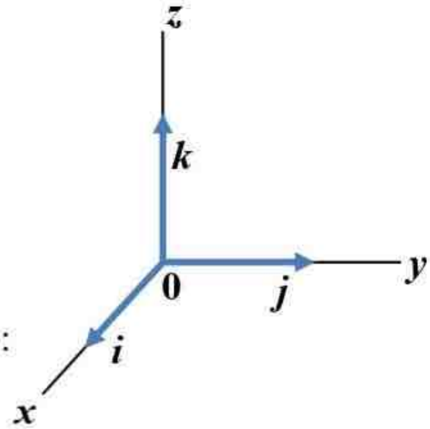

The magnitude of a vector $\boldsymbol{A}$, denoted by $|\vec{A}|$ or by $\boldsymbol{A}$, is defined as:

$$\boldsymbol{A} = |\vec{A}| = [A_x^2 + A_y^2 + A_z^2]^{1/2}$$

Geometrically, the magnitude of a vector is its length, that is, the length of the diagonal of the rectangular parallelepiped whose sides are $A_x$, $A_y$, $A_z$.

من الناحية الهندسية ، فإن مقدار المتجه يمثل طوله ، أي طول قطر متوازي المستطيلات الذي جوانبه هي مركبات المتجه

- **Unit Coordinate Vectors**




A unit vector is a vector whose magnitude is *unity*. There are three unit Coordinate vectors which also called *basis* vectors.

$\boldsymbol{i} = [1,0,0]$ , $\boldsymbol{j} = [0,1,0]$ , $\boldsymbol{k} = [0,0,1]$

Any vector can be expressed in terms of these vectors:

$$\vec{A} = \boldsymbol{i}A_x + \boldsymbol{j}A_y + \boldsymbol{k}A_z$$

متجه الوحدة هو المتجه الذي مقداره وحدة واحدة

**Example:**

Find the sum and the magnitude of the sum of the two vectors $A = (1,0,2)$ and $B = (0,1,1)$.

**Solution:**

$$\vec{A} + \vec{B} = (1,0,2) + (0,1,1) = (1+0,\ 0+1, 2+1) = (1,1,3).$$

$$|\vec{A} + \vec{B}| = (1+1+9)^{1/2} = \sqrt{11}$$

## 1.3 Vector products

### 1.3.1 The Scalar Product

Given two vectors $\vec{A}$ and $\vec{B}$, the scalar product or "*dot*" product, $(\vec{A}.\vec{B})$, is the scalar defined by the equation:

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

***Laws of scalar product algebra***

- $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ Commutation law
- $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$ Distribution law
- $(C\vec{A})\cdot\vec{B} = C(\vec{A}\cdot\vec{B})$ Multiplication by scaler

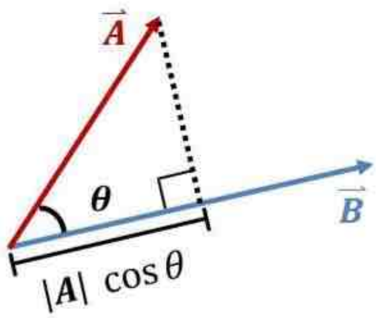
$\vec{A}.\vec{B}$ equal to the length of the projection of $(\vec{A})$ on $(\vec{B})$ times the length $(\vec{B})$.

$\vec{A}\cdot\vec{B}$ يساوي طول المسقط من $\vec{A}$ على $\vec{B}$ مضروب بطول $(\vec{B})$

From analytical geometry the cosine of the angle between two lines given as:

$$\cos\theta = \frac{A_xB_x+A_yB_y+A_zB_z}{\left(A_x^2+A_y^2+A_z^2\right)^{\frac{1}{2}}\cdot\left(B_x^2+B_y^2+B_z^2\right)^{\frac{1}{2}}} = \frac{\vec{A}\cdot\vec{B}}{AB}$$

$$\therefore \vec{A}\cdot\vec{B} = AB\cos\theta$$




***Properties of the Scalar Product***

1. $\vec{A}\cdot\vec{A} = |\vec{A}|^2 = A^2$

2. $\vec{A}\cdot\vec{B} = 0 \;\Rightarrow \vec{A}\perp\vec{B} \qquad i.e \qquad \theta = 90^o$

3. If $(\boldsymbol{i},\ \boldsymbol{j},\boldsymbol{k})$ is an orthonormal basis then :

$$\left.\begin{aligned}\boldsymbol{i}\cdot\boldsymbol{i} &= \boldsymbol{j}\cdot\boldsymbol{j} = \boldsymbol{k}\cdot\boldsymbol{k} = \mathbf{1}\\ \boldsymbol{i}\cdot\boldsymbol{j} &= \boldsymbol{j}\cdot\boldsymbol{k} = \boldsymbol{k}\cdot\boldsymbol{i} = \mathbf{0}\end{aligned}\right\}$$

**(H.W) Prove that**

4. If $\vec{A} = \boldsymbol{i}A_x + \boldsymbol{j}A_y + \boldsymbol{k}A_z$

$$\vec{B} = \boldsymbol{i}B_x + \boldsymbol{j}B_y + \boldsymbol{k}B_z$$

$$\therefore \vec{A}\cdot\vec{B} = \left(\boldsymbol{i}A_x + \boldsymbol{j}A_y + \boldsymbol{k}A_z\right)\cdot(\boldsymbol{i}B_x + \boldsymbol{j}B_y + \boldsymbol{k}B_z)$$

$$\vec{A}\cdot\vec{B} = \boldsymbol{i}\cdot\boldsymbol{i}\,A_xB_x + \boldsymbol{i}\cdot\boldsymbol{j}A_xB_y + \boldsymbol{i}\cdot\boldsymbol{k}A_xB_z + \boldsymbol{j}\cdot\boldsymbol{i}\,A_yB_x + \boldsymbol{j}\cdot\boldsymbol{j}A_yB_y + \boldsymbol{j}\cdot\boldsymbol{k}A_yB_z + \boldsymbol{k}\cdot\boldsymbol{i}\,A_zB_x + \boldsymbol{k}\cdot\boldsymbol{j}A_zB_y + \boldsymbol{k}\cdot\boldsymbol{k}A_zB_z$$

$$\therefore \vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

**Example:**

Given the two vectors $\vec{A} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ , $\vec{B} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$, find $\vec{A}\cdot\vec{B}$

**Solution:**

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

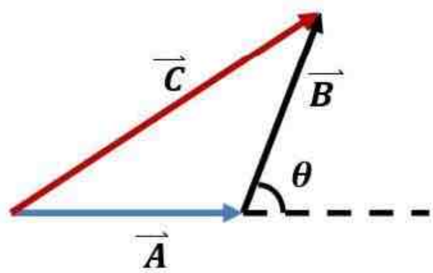
$$\vec{A}\cdot\vec{B} = 2*1 + 1*(-1) + (-1*2)$$

$$\vec{A}\cdot\vec{B} = 2 - 1 - 2 = -1$$

**Some Application of Dot Product**

**1.** ***Low of Cosines***

If $\vec{A}.\vec{B}$ and $\vec{C}$ are the sides of a triangular, then:




$$\vec{C} = \vec{A} + \vec{B}$$

$$\vec{C}\cdot\vec{C} = \left(\vec{A}+\vec{B}\right)\cdot\left(\vec{A}+\vec{B}\right)$$

$$= \vec{A}\cdot\vec{A} + \vec{B}\cdot\vec{A} + \vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{B}$$

$$= \vec{A}\cdot\vec{A} + 2\vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{B}$$

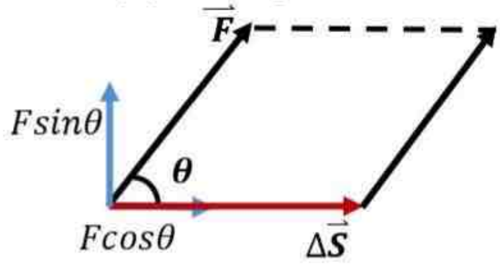
$$= \left|\vec{A}^2\right| + 2\vec{A}.\vec{B} + \left|\vec{B}^2\right|$$

$$\therefore C^2 = A^2 + 2\vec{A}\cdot\vec{B} +$$

$$C^2 = A^2 + B^2 + 2AB\cos\theta$$

**2.** ***Work***

Suppose an object under the action of a constant force $(\vec{F})$ undergoes a linear displacement $(\Delta\vec{S})$, as shown:




$$\Delta W = F\cos\theta\,\Delta S$$

where $\theta$: is the angle between $\vec{F}$ and $\Delta\vec{S}$

$$\Delta W = \vec{F}\cdot\Delta\vec{S}$$

**Example:**

If $\vec{A} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and $\vec{B} = 4\boldsymbol{i} - 3\boldsymbol{k}$, find the magnitude of $\vec{A}$ and $\vec{B}$ and the angle between them.

**Solution:**

$$|\vec{A}| = A = (A_x^2 + A_y^2 + A_z^2)^{1/2} = [2^2 + (-1)^2 + 2^2]^{1/2} = \sqrt{9} = 3$$

$$|\vec{B}| = B = (B_x^2 + B_y^2 + B_z^2)^{1/2} = [4^2 + (-3)^2]^{1/2} = \sqrt{25} = 5$$

$$\vec{A} \cdot \vec{B} = (2 * 4) + (-1 * 0) + (2 * -3) = 8 + 0 - 6 = 2$$

$$\therefore \vec{A} \cdot \vec{B} = AB\cos\theta$$

$$\therefore \cos\theta = \frac{\vec{A}.\vec{B}}{AB} = \frac{2}{3*5} = \frac{2}{15}$$

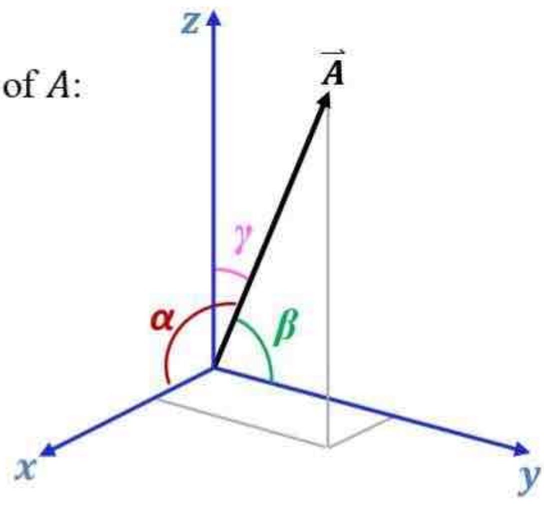
$$\therefore \theta = cos^{-1}\left(\frac{2}{15}\right) = 82.3°$$

**Expressing any Vector as the Product of its Magnitude by a Unit Vector: Projection** التعبير عن المتجه كحاصل ضرب لمقداره في متجه الوحدة: المسقط

$$\vec{A} = \boldsymbol{i}A_x + \boldsymbol{j}A_y + \boldsymbol{k}A_z$$

Multiply and divide on the right by the magnitude of $A$:

$$\therefore \vec{A} = \left(\boldsymbol{i}\frac{A_x}{A} + \boldsymbol{j}\frac{A_y}{A} + \boldsymbol{k}\frac{A_z}{A}\right)A$$




The ***direction cosines*** of the vector $\vec{A}$ are:

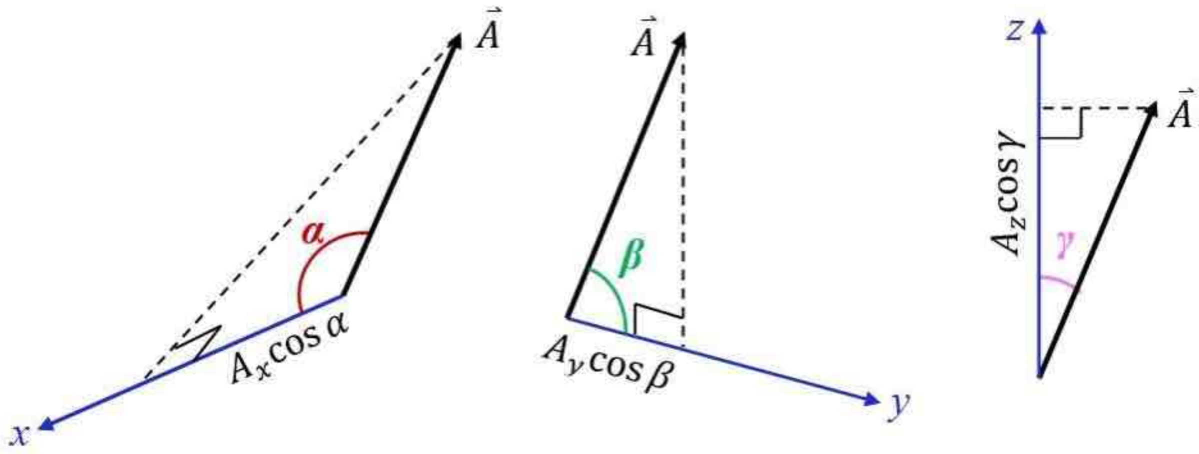
$$\left.\begin{aligned} \cos\alpha &= \frac{A_x}{A} \\ \cos\beta &= \frac{A_y}{A} \\ \cos\gamma &= \frac{A_z}{A} \end{aligned}\right\} \textbf{\textit{Direction cosines}}$$

where: α, β, and γ are the ***direction angles***.

$\alpha$ is the angle between $\vec{A}$ and the $x$-axis

$\beta$ is the angle between $\vec{A}$ and the $y$-axis

$\gamma$ is the angle between $\vec{A}$ and the $z$-axis




$\therefore \vec{A} = A(\boldsymbol{i}\cos\alpha + \boldsymbol{j}\cos\beta + \boldsymbol{k}\cos\gamma) = A(\cos\alpha + \cos\beta + \cos\gamma)$

Or $\vec{A} = A\vec{\boldsymbol{n}}$

where $\vec{\boldsymbol{n}}$ is a unit vector whose components are $\cos\alpha$, $\cos\beta$, and $\cos\gamma$

Consider any other vector $\vec{B}$. Then, the projection of $\vec{B}$ on $\vec{A}$ is:

$$B\cos\theta = \frac{\vec{B}\cdot\vec{A}}{A} = \vec{B}\cdot\vec{n}$$

where $\theta$ is the angle between $\vec{A}$ and $\vec{B}$.

**Example:**

If $\vec{A} = 2\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}$ is a vector, find the unit vector of $\vec{A}$ and the direction cosines and direction angles.

**Solution:**

$|\vec{A}| = A = [2^2 + (-3)^2 + 4^2]^{1/2} = \sqrt{29}$

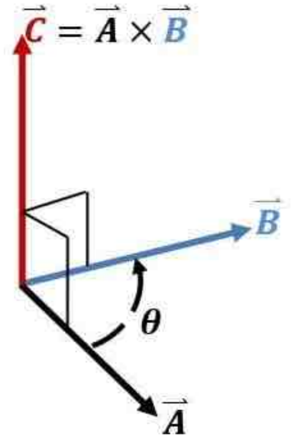
$\vec{n} = \frac{\vec{A}}{A} = \frac{2\boldsymbol{i}-3\boldsymbol{j}+4\boldsymbol{k}}{\sqrt{29}} = \frac{2\boldsymbol{i}}{\sqrt{29}} - \frac{3\boldsymbol{j}}{\sqrt{29}} + \frac{4\boldsymbol{k}}{\sqrt{29}}$

Direction cosines $= (\frac{2}{\sqrt{29}}, -\frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}})$

$$\left.\begin{aligned}
\cos\alpha &= \frac{2}{\sqrt{29}} &&\Rightarrow \alpha = \cos^{-1}\left(\frac{2}{\sqrt{29}}\right) = 68.17^\circ \\
\cos\beta &= -\frac{3}{\sqrt{29}} &&\Rightarrow \beta = \cos^{-1}\left(-\frac{3}{\sqrt{29}}\right) = 123.89^\circ \\
\cos\gamma &= \frac{4}{\sqrt{29}} &&\Rightarrow \gamma = \cos^{-1}\left(\frac{4}{\sqrt{29}}\right) = 41.97^\circ
\end{aligned}\right\} \textbf{\textit{Direction cosines}}$$

### 1.3.2 Vector Product (Cross Product)

If $\vec{A} \cdot \vec{B}$ two vectors, the vector product $(\vec{A} \times \vec{B})$ defined by:




$$\vec{A} \times \vec{B} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

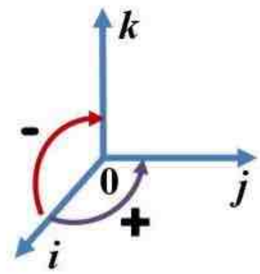
$$\vec{A} \times \vec{B} = \boldsymbol{i} \begin{vmatrix} A_y & A_z \\ B_z & B_z \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$\therefore \vec{A} \times \vec{B} = \boldsymbol{i}\left(A_y B_z - A_z B_y\right) + \boldsymbol{j}(A_z B_x - A_x B_z) + \boldsymbol{k}\left(A_x B_y - A_y B_x\right)$$

$$or\ \vec{A} \times \vec{B} = [\left(A_y B_z - A_z B_y\right), (A_z B_x - A_x B_z), \left(A_x B_y - A_y B_x\right)]$$

$$\boxed{\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = 0}$$

$$\boxed{\begin{array}{lcl} \boldsymbol{j} \times \boldsymbol{k} = -\boldsymbol{k} \times \boldsymbol{j} = \boldsymbol{i} & , & \boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i} \\ \boldsymbol{k} \times \boldsymbol{i} = -\boldsymbol{i} \times \boldsymbol{k} = \boldsymbol{j} & , & \boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j} \\ \boldsymbol{i} \times \boldsymbol{j} = -\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{k} & , & \boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k} \end{array}}$$




***Proof***

$$\boldsymbol{i} \times \boldsymbol{j} = (1,0,0) \times (0,1,0)$$
$$= (0-0), (0-0), (1-0)$$
$$= (0,0,1)$$
$$\therefore \boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$$

$$\left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right| \left|\vec{B}\right| \sin\theta$$

or $\vec{A} \times \vec{B} = \left(\left|\vec{A}\right| \left|\vec{B}\right| \sin\theta\right) \cdot \vec{n}$

where $(0 \le \theta \le 180)$ is the angle between $\vec{A}$ and $\vec{B}$

$\vec{n}$ is the ***unit vector normal*** to the plane containing the two vector $\vec{A}$ and $\vec{B}$.

$$\vec{n} = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$$

***Properties of the Vector Product***

1. $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$
2. $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$
3. $C(\vec{A}\times\vec{B}) = (C\vec{A})\times\vec{B} = \vec{A}\times(C\vec{B})$
4. $\vec{A}\times\vec{B} = 0$ if ( and only if ) $\vec{A}$ and $\vec{B}$ are parallel or one of them is zero.
5. $\vec{A}\times\vec{A} = 0$

**Example:**

Given the two vectors $\vec{A} = 2\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}$ , $\vec{B} = \boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$, find $\vec{A}\times\vec{B}$.

**Solution:**

$$\vec{A}\times\vec{B} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$\vec{A}\times\vec{B} = \boldsymbol{i}(2-1)+\boldsymbol{j}(-1-4)+\boldsymbol{k}(-2-1)$$

$$\vec{A}\times\vec{B} = \boldsymbol{i}-5\boldsymbol{j}-3\boldsymbol{k}$$

**Applications of the Vector Product**

- ***Torque***

Let $(\vec{F})$ be a force act at a point $P(x,y,z)$ and $(\vec{r})$ is a vector represented by:

$\overrightarrow{OP} = \vec{r} = \boldsymbol{i}x+\boldsymbol{j}y+\boldsymbol{k}z$

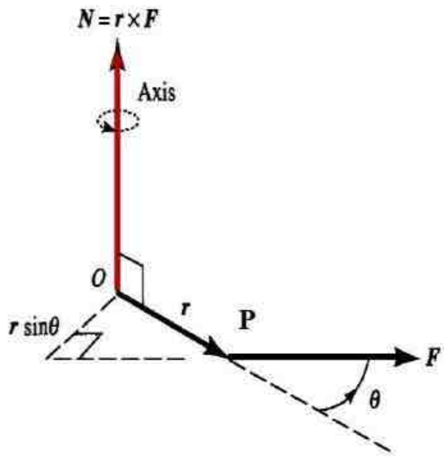
Then the moment (Torque) ($N$) about a given point (O) is defined as the cross product of $(\vec{F})$ and $(\vec{r})$

$$\vec{N} = \vec{r}\times\vec{F}$$

* *The magnitude of the torque is*

$|\vec{N}| = |\vec{r} \times \vec{F}| = rF \sin\theta$

$\theta$ is the angle between $\vec{r}$ and $\vec{F}$




* For several forces applied on a single body at different points, the moment add vectorially:

$\sum_i (\vec{r_i} \times \vec{F_i}) = \sum_i \vec{N_i}$

* The condition for rotational equilibrium is that the vector sum of all the moments is zero.

شرط الاتزان الدوراني ان يكون مجموع العزوم يساوي صفر

$\sum_i (\vec{r_i} \times \vec{F_i}) = \sum_i \vec{N_i} = 0$

## The Triple Product

### 1. The Triple Scalar Product

It is defined by: $\vec{A} \cdot (\vec{B} \times \vec{C})$

and its value is a scalar.

***Its properties are:***

**1.** $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$

The dot and cross product ***may be interchanged*** in the triple scalar product

الضرب العددي والاتجاهي يتغير في حالة الضرب الثلاثي

**2.** $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$ if one of them is zero

**3.** If $\vec{A} = \boldsymbol{i}A_x + \boldsymbol{j}A_y + \boldsymbol{k}A_z$

$\vec{B} = \boldsymbol{i}B_x + \boldsymbol{j}B_y + \boldsymbol{k}B_z$

$\vec{C} = \boldsymbol{i}C_x + \boldsymbol{j}C_y + \boldsymbol{k}C_z$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

## 2. The Triple Vector Product

It is defined by $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)$, its value is a vector.

***Its properties is:***

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\left(\vec{A}\cdot\vec{C}\right)\vec{B}-\left(\vec{A}\cdot\vec{B}\right)\vec{C}$$

The triple vector product is ***anti-commutative and not associative***.

الضرب الثلاثي للمتجهات لايكون متبادل وغير قابل للتجميع

## 1.4 Derivation of Vector

If the vector $(\vec{A})$ is a function of a single scalar variable $u$ then it components, are also functions of $(u)$. The vector may represent position, velocity, and so on.

اذا كان المتجه دالة لمتغير عددي عندها تكون كل مركبة فيه دالة لذلك المتغير. وقد يكون هذا المتجه موقع او سرعة

$$\vec{A}(u)=A_x(u)\boldsymbol{i}+A_y(u)\boldsymbol{j}+A_z(u)\boldsymbol{k}$$

$$\frac{d\vec{A}}{du}=\lim_{\Delta u\to 0}\frac{\Delta\vec{A}}{\Delta u}=\lim_{\Delta u\to 0}\left[\boldsymbol{i}\frac{dA_x}{du}+\boldsymbol{j}\frac{dA_y}{du}+\boldsymbol{k}\frac{dA_z}{du}\right]$$

$$\therefore\frac{d\vec{A}}{du}=\boldsymbol{i}\frac{dA_x}{du}+\boldsymbol{j}\frac{dA_y}{du}+\boldsymbol{k}\frac{dA_z}{du}$$

***Properties***

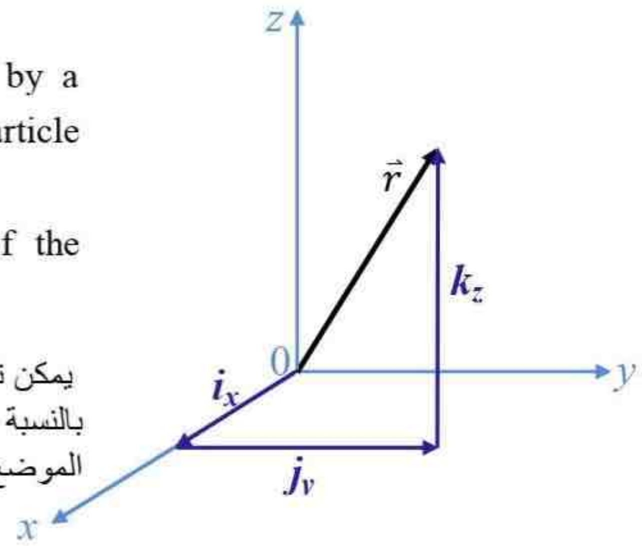
**1.** $\frac{d}{du}(\vec{A}+\vec{B})=\frac{d\vec{A}}{du}+\frac{d\vec{B}}{du}=\dot{\vec{A}}+\dot{\vec{B}}$

**2.** $\frac{d}{du}(C\vec{A})=\frac{dC}{du}\vec{A}+C\frac{d\vec{A}}{du}$

**3.** $\frac{d}{du}\left(\vec{A}\cdot\vec{B}\right)=\frac{d\vec{A}}{du}\cdot\vec{B}+\vec{A}\cdot\frac{d\vec{B}}{du}=\dot{\vec{A}}\cdot\vec{B}+\vec{A}\cdot\dot{\vec{B}}$

**4.** $\frac{d}{du}\left(\vec{A}\times\vec{B}\right)=\frac{d\vec{A}}{du}\times\vec{B}+\vec{A}\times\frac{d\vec{B}}{du}=\dot{\vec{A}}\times\vec{B}+\vec{A}\times\dot{\vec{B}}$

## 1.5 The Position Vector of a Particle




The position of a particle can be specified by a single vector, the displacement of the particle relative to the origin of the coordinate system.

This vector is called the ***position vector*** of the particle. Given as:

يمكن تحديد موضع الجسيم بواسطة متجه واحد، وهو إزاحة الجسيم بالنسبة إلى نقطة الأصل لنظام الإحداثيات. وهذا المتجه يدعى متجه الموضع

$$\vec{r} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$$

where:

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

The value of $\vec{r}$ is:

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2} = S$$

The magnitude of displacement is called ***distance***

مقدار متجه الموضع يسمى المسافة

## 1.6 The Velocity Vector

The derivative of $\vec{r}$ with respect to $t$ is called ***velocity***:

مشتقة متجه الموضع بالنسبة للزمن تسمى السرعة

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(\boldsymbol{i}x) + \frac{d}{dt}(\boldsymbol{j}y) + \frac{d}{dt}(\boldsymbol{k}z)$$

$$\vec{v} = \boldsymbol{i}\frac{dx}{dt} + x\frac{d\boldsymbol{i}}{dt} + \boldsymbol{j}\frac{dy}{dt} + y\frac{d\boldsymbol{j}}{dt} + \boldsymbol{k}\frac{dz}{dt} + z\frac{d\boldsymbol{k}}{dt}$$

$$\frac{d\boldsymbol{i}}{dt}, \quad \frac{d\boldsymbol{j}}{dt}, \quad \frac{d\boldsymbol{k}}{dt} = 0 \text{ in Cartesian coordinate only}$$

$$\vec{v} = \boldsymbol{i}\frac{dx}{dt} + \boldsymbol{j}\frac{dy}{dt} + \boldsymbol{k}\frac{dz}{dt}$$

$$\boxed{\begin{aligned}\vec{v} &= \boldsymbol{i}\dot{x} + \boldsymbol{j}\dot{y} + \boldsymbol{k}\dot{z} \\ \vec{v} &= \boldsymbol{i}v_x + \boldsymbol{j}v_y + \boldsymbol{k}v_z\end{aligned}}$$

***Velocity vector***

The magnitude of velocity is called ***Speed:***

$$\therefore v = |\vec{v}| = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}$$

مقدار متجه السرعة يسمى الانطلاق

## 1.7 The Acceleration Vector

The time derivative of the velocity is called the ***acceleration***:

مشتقة متجه السرعة بالنسبة للزمن يسمى التعجيل

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{\boldsymbol{d}}{\boldsymbol{dt}}(\boldsymbol{i}\dot{x} + \boldsymbol{j}\dot{y} + \boldsymbol{k}\dot{z})$$

$$= \boldsymbol{i}\frac{d\dot{x}}{dt} + \boldsymbol{j}\frac{d\dot{y}}{dt} + \boldsymbol{k}\frac{d\dot{z}}{dt})$$

$$\vec{a} = \boldsymbol{i}\ddot{x} + \boldsymbol{j}\ddot{y} + \boldsymbol{k}\ddot{z}$$ ***The acceleration vector***

**Example:**

The position vector of a particle at time $t$ is given by $\vec{r} = (2t^2 - 5t)\boldsymbol{i} + (4t + 2)\boldsymbol{j} + t^3\boldsymbol{k}$ , find velocity and acceleration vectors of the particle at time $t$.

**Solution:**

$$\vec{v} = \frac{d\vec{r}}{dt} = (4t - 5)\boldsymbol{i} + 4\boldsymbol{j} + 3t^2\boldsymbol{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = 4\boldsymbol{i} + 6t\boldsymbol{k}$$

## 1.8 Vector Integral

$$\because \vec{a} = \frac{d\vec{v}}{dt}$$

$$\int d\vec{v} = \int \vec{a}\, dt$$

$$\therefore \vec{v} = \int \vec{a}\, dt \qquad \dots\dots\dots (1)$$

$$\because \vec{v} = \frac{d\vec{r}}{dt}$$

$$\int d\vec{r} = \int \vec{v}\, dt$$

$$\therefore \vec{r} = \int \vec{v}\, dt \qquad \dots\dots\dots (2)$$

**Example:**

A particle moves along the x-axis at time *t* with acceleration given by:

$\vec{a} = 12t^2 - 6t + 6$ , Find the velocity and displacement.

**Solution:**

$$\vec{a} = \frac{d\vec{v}}{dt} \quad \rightarrow \quad \int d\vec{v} = \int \vec{a}\, dt$$

$$\therefore \vec{v} = \int (12t^2 - 6t + 6)dt = \frac{12}{3}t^3 - \frac{6}{2}t^2 + 6t$$

$$\vec{v} = 4t^3 - 3t^2 + 6t$$

$$\because \vec{v} = \frac{d\vec{r}}{dt} \quad \rightarrow \quad \int d\vec{r} = \int \vec{v}\, dt$$

$$\therefore \vec{r} = \int (4t^3 - 3t^2 + 6t)dt = \frac{4}{4}t^4 - \frac{3}{3}t^3 + \frac{6}{2}t^2$$

$$\vec{r} = t^4 - t^3 + 3t^2$$

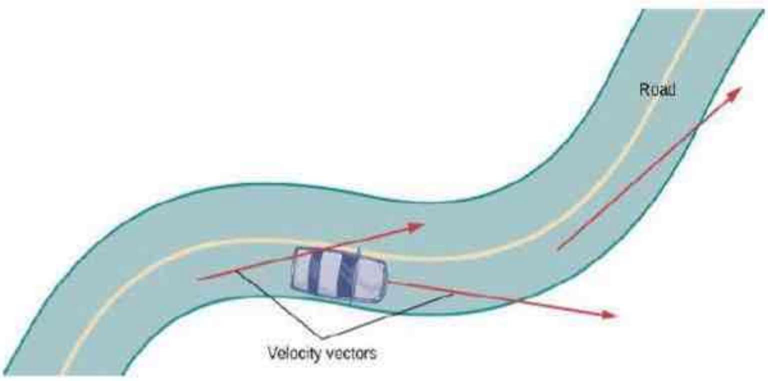
### 1.9 Tangential and Normal Components of Acceleration

When a particle moves along a curved path, it is sometimes convenient to describe its motion using coordinates other than Cartesian. When the path of motion is known, normal ($\boldsymbol{n}$) and tangential ($\boldsymbol{\tau}$) coordinates are often used.

عندما يتحرك الجسيم على طول مسار منحني ، يكون من المناسب وصف حركته باستخدام إحداثيات أخرى غير الديكارتية. عندما يكون مسار الحركة معروفًا ، يتم استخدام الإحداثي العمودي $(n)$ و الاحداثي المماسي $(\tau)$.

$$\vec{v} = v\boldsymbol{\tau}$$

where $\boldsymbol{\tau}$ is ***unit tangent vector***




$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv\boldsymbol{\tau}}{dt}$$

$$\vec{a} = \frac{dv}{dt}\boldsymbol{\tau} + v\frac{d\boldsymbol{\tau}}{dt}$$

$$\frac{d\boldsymbol{\tau}}{dt} = \boldsymbol{n}$$

For small $\psi$

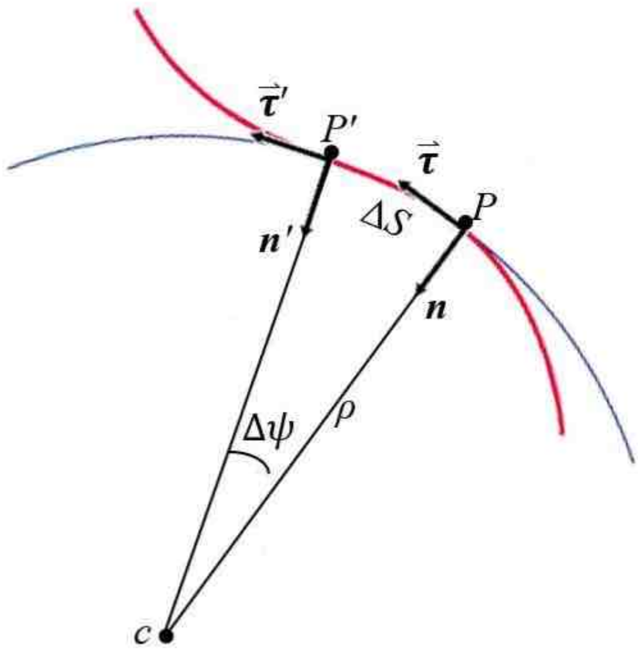
$$\Delta\boldsymbol{\tau} \approx \Delta\psi \approx 0$$

$\Delta\boldsymbol{\tau} \perp \boldsymbol{\tau}$

$\frac{\Delta\tau}{\Delta\psi} = \mathbf{1}$

$\frac{\Delta\boldsymbol{\tau}}{\Delta\psi} \perp \boldsymbol{\tau}$

$\frac{\Delta\boldsymbol{\tau}}{\Delta\psi} = \boldsymbol{n}$ ***Unit normal vector***




Using ***chain rule***

$$\frac{d\boldsymbol{\tau}}{dt} = \frac{d\boldsymbol{\tau}}{d\psi}\frac{d\psi}{dt} = \boldsymbol{n}\frac{d\psi}{dt}$$

$$\frac{d\boldsymbol{\tau}}{dt} = \boldsymbol{n}\frac{d\psi}{dt}\frac{ds}{ds} = \boldsymbol{n}\frac{d\psi}{ds}\frac{ds}{dt}$$

$$\frac{d\boldsymbol{\tau}}{dt} = \boldsymbol{n}\frac{v}{\rho}$$

***where*** $\vec{v} = \frac{ds}{dt}$ , $\frac{1}{\rho} = \frac{d\psi}{ds}$

ρ is radius of curvature

$$\vec{a} = \frac{dv}{dt}\boldsymbol{\tau} + \frac{v^2}{\rho}\boldsymbol{n}$$

$\vec{a} = \dot{v}\boldsymbol{\tau} + \frac{v^2}{\rho}\boldsymbol{n}$ ***Equation of acceleration*** معادلة التعجيل

$|\vec{a}| = (\dot{v}^2 + \frac{v^4}{\rho^2})^{1/2}$ ***Magnitude of total acceleration*** مقدار التعجيل الكلي

$a_\tau = \dot{v} = \ddot{s}$ ***Tangential acceleration*** التعجيل المماسي

$a_n = \frac{v^2}{\rho}$ ***Centripetal acceleration*** تعجيل قوة الجذب المركزي

### Special Cases of Motion

- When the particle moves along a ***straight line***.

$$\rho \to \infty \quad \Rightarrow \quad a_n = \frac{v^2}{\rho} = 0 \quad \Rightarrow \quad a = a_\tau = \dot{v}$$

The tangential component represents the time rate of change in the magnitude of the velocity.

عند حركة الجسم على خط مستقيم فأن المركبة المماسية للتعجيل تمثل المعدل الزمني لتغير مقدار السرعة في حين ان المركبة العمودية تساوي صفر

- When the particle moves along a curve at ***constant speed***.

$$a_\tau = \dot{v} = 0 \quad \Rightarrow \quad a = a_n = \frac{v^2}{\rho}$$

The normal component represents the time rate of change in the direction of the velocity. **(*Centripetal acceleration*)**

عند حركة الجسم على منحني بسرعة ثابتة فان المركبة العمودية للتعجيل تمثل المعدل الزمني للتغير في مقدار السرعة (تعجيل مركزي) في حين ان التعجيل المماسي يساوي صفر

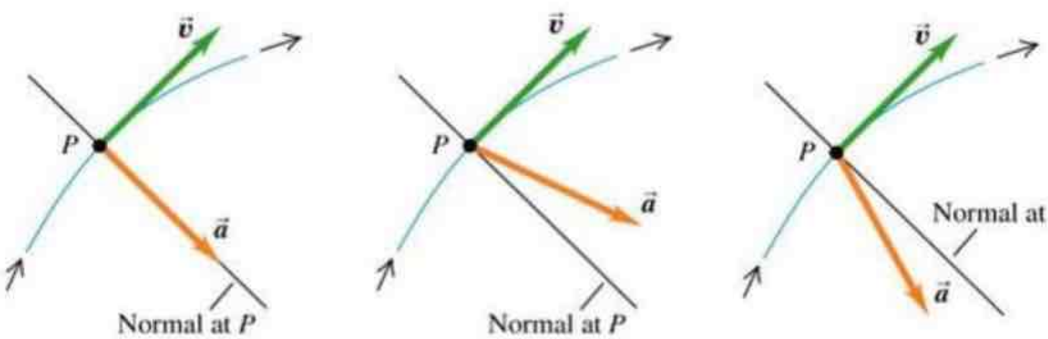
- When the particle moves along a curve at ***non constant speed***

  - At the increasing speed with certain rate $\dot{v}$, the acceleration is away from the center.

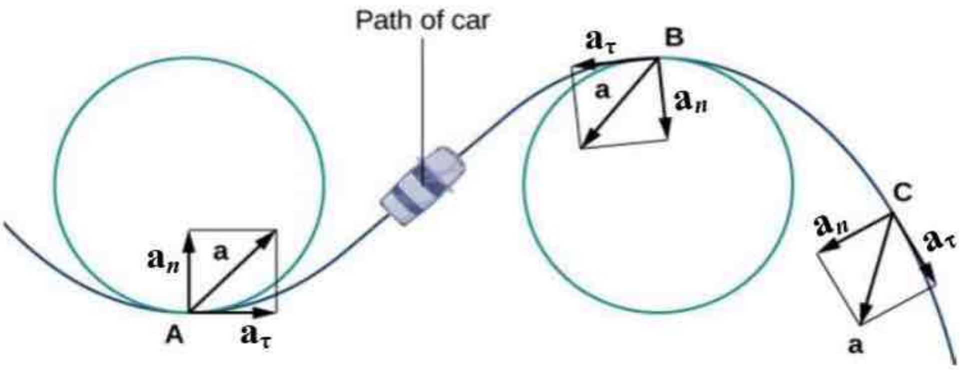
    عند حركة الجسم على منحني بسرعة غير ثابتة فان كانت هناك زيادة في معدل السرعة عندها التعجيل يكون بعيدا عن المركز نحو الاتجاه الامامي

  - At the decreasing speed with certain rate $\dot{v}$, the acceleration is in the opposite direction.

عند حركة الجسم على منحني بسرعة غير ثابتة فان كانت هناك تناقص في معدل السرعة عندها التعجيل ايضا يكون بعيدا عن المركز لكن بالاتجاه المعاكس




**(a)** Constant speed **(b)** Increasing speed **(c)** Decreasing speed

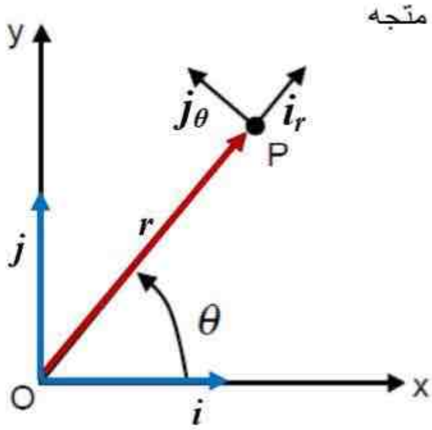


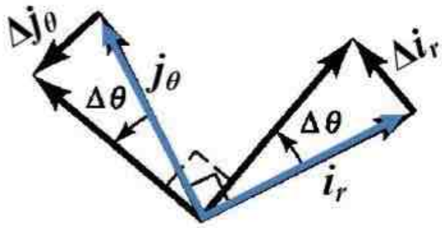

### 1.10 Velocity and Acceleration in Plan Polar Coordinates

It is often convenient to employ polar coordinates $\boldsymbol{r}$, θ to express the position of a particle moving in a plane. Vectorially, the position of the particle can be written as the product of the radial distance $r$ by a unit radial vector $\boldsymbol{i}_r$:

$$\vec{r} = r\boldsymbol{i}_r \qquad \ldots\ldots\ldots (1)$$

غالبًا ما يكون من المناسب استخدام إحداثيات قطبية لوصف موضع الجسيم المتحرك في المستوى. من الناحية الحركية ، يمكن كتابة موضع الجسيم كناتج لحاصل الضرب المسافة القطرية r في متجه الوحدة القطري




As the particle moves, both $\boldsymbol{r}$ and $\boldsymbol{i}_r$ vary

$$\therefore \vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\boldsymbol{i}_r) = \left(\frac{dr}{dt}\right)\boldsymbol{i}_r + r\left(\frac{d\boldsymbol{i}_r}{dt}\right)$$

$$\vec{v} = \dot{r}\boldsymbol{i}_r + r\left(\frac{d\boldsymbol{i}_r}{dt}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

To obtain the polar formula for the velocity of $P$, we differentiate eqn. (1) with respect to $(t)$.

Hence by the chain rule:

$$\frac{d\boldsymbol{i}_r}{dt} = \frac{\partial \boldsymbol{i}_r}{\partial\theta} \times \frac{\partial\theta}{\partial t} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Any vector $\vec{A}$ in the plan $(\boldsymbol{i}\,,\boldsymbol{j})$ can be expanded in the form:

$\vec{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j}$

where $A_x\,, A_y$ are the component of $\vec{A}$ in the $(\boldsymbol{i})$ and $(\boldsymbol{j})$ direction respectively.

We will now evaluate the two derivatives $\frac{d\boldsymbol{i}_r}{d\theta}\,, \frac{d\boldsymbol{j}_r}{d\theta}$ , we expand $\boldsymbol{i}_r\,,\boldsymbol{j}_\theta$ in terms of Cartesian basis vectors $(\boldsymbol{i}\,,\boldsymbol{j})$ this gives

$$\boldsymbol{i}_r = \cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$\boldsymbol{j}_\theta = -\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

$$\frac{\partial\boldsymbol{i}_r}{\partial\theta} = -\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j}$$

$$\frac{\partial\boldsymbol{i}_r}{\partial\theta} = \boldsymbol{j}_\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots. (6)$$

$$\frac{\partial\boldsymbol{j}_\theta}{\partial\theta} = -[\cos\theta\,\boldsymbol{i} - \sin\theta\,\boldsymbol{j}]$$

$$\frac{\partial\boldsymbol{j}_\theta}{\partial\theta} = -\boldsymbol{i}_r \quad \ldots\ldots\ldots\ldots\ldots\ldots.. (7)$$

$$\therefore \frac{d\boldsymbol{i}_r}{dt} = \boldsymbol{j}_\theta \times \dot{\boldsymbol{\theta}} = \dot{\theta}\,\boldsymbol{j}_\theta \quad \ldots\ldots\ldots.. (8)$$

$$\vec{v} = \dot{r}\boldsymbol{i}_r + (r\dot{\theta})\,\boldsymbol{j}_\theta \quad \ldots\ldots\ldots.. (9)$$ ***Velocity in polar coordinate***

The velocity of $P$ is the vector sum of an outward radial velocity $(\dot{r})$ and a transverse velocity $(r\dot{\theta})$

To obtain the acceleration we differential $\vec{V}$ with respect to $t$:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{r}\boldsymbol{i}_r) + \frac{d}{dt}\left[(r\dot{\theta})\boldsymbol{j}_\theta\right]$$

$$= \ddot{r}\boldsymbol{i}_r + \dot{r}\frac{d\boldsymbol{i}_r}{dt} + \left(\dot{r}\dot{\theta} + r\ddot{\theta}\right)\boldsymbol{j}_\theta + (r\dot{\theta})\frac{d\boldsymbol{j}_\theta}{dt}$$

$$= \ddot{r}\boldsymbol{i}_r + \dot{r}\left(\frac{d\boldsymbol{i}_r}{d\theta} \times \frac{d\theta}{dt}\right) + \left(\dot{r}\dot{\theta} + r\ddot{\theta}\right)\boldsymbol{j}_\theta + (r\dot{\theta})\left(\frac{d\boldsymbol{j}_\theta}{d\theta} \times \frac{d\theta}{dt}\right)$$

$$= \ddot{r}\boldsymbol{i}_r + (\dot{r}\dot{\theta})\boldsymbol{j}_\theta + \left(\dot{r}\dot{\theta} + r\ddot{\theta}\right)\boldsymbol{j}_\theta + (r\dot{\theta})\left(-\boldsymbol{i}_r\dot{\theta}\right)$$

$$= \ddot{r}\boldsymbol{i}_r + (\dot{r}\dot{\theta})\boldsymbol{j}_\theta + \left(\dot{r}\dot{\theta}\right)\boldsymbol{j}_\theta + (r\ddot{\theta})\boldsymbol{j}_\theta - \left(r\dot{\theta}^2\right)\boldsymbol{i}_r$$

$$\therefore \vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\boldsymbol{i}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\boldsymbol{j}_\theta \quad \ldots\ldots\ldots\ldots (10)$$ ***Acceleration in polar coordinate***

$$a_r = \left(\ddot{r} - r\dot{\theta}^2\right)$$ ***The magnitude of the radial component of the acceleration***

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$$ ***The magnitude of the transverse component of the acceleration***

If a particle moves on a ***circle*** of constant radius b,

$a_r = b\dot{\theta}^2 \quad , \quad a_\theta = b\ddot{\theta}$

If the particle moves along a ***fixed radial line***

$a_r = \ddot{r} \quad , \quad a_\theta = 0$

**Example:**

A particle moves on a spiral path such that the position in polar coordinate is given by $r = bt^2, \theta = ct$ , where $c$ and $d$ are constants , find the velocity and acceleration as a function of time.

**Solution:**

$r = bt^2 \ , \ \dot{r} = 2bt \ , \ddot{r} = 2b$

$$\begin{aligned}\vec{v} &= \dot{r}\boldsymbol{i_r} + (r\dot{\theta})\,\boldsymbol{j_\theta} \\ &= \boldsymbol{i_r}\frac{d}{dt}(bt^2) + \boldsymbol{j_\theta}(bt^2)\frac{d}{dt}(ct) \\ &= \boldsymbol{i_r}(2bt^2) + \boldsymbol{j_\theta}(bt^2c)\end{aligned}$$

$\theta = ct \ \ , \dot{\theta} = c \ , \ddot{\theta} = 0$

$$\begin{aligned}\vec{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\boldsymbol{i_r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\boldsymbol{j_\theta} \\ &= (2b - bt^2c^2)\boldsymbol{i_r} + (bt^2(0) + 2(2bt)c)\boldsymbol{j_\theta} \\ &= (2b - bt^2c^2)\boldsymbol{i_r} + (4btc)\boldsymbol{j_\theta}\end{aligned}$$

**Example:**

A particle is sliding along a radial groove in a rotating turn able has polar coordinate at time $t$ is given by $r = ct$ , $\theta = gt$ , where $c$ and $g$ are constants , find the velocity and acceleration vectors of the particle at time $t$ and the speed of the particle at time $t$ .

Deduce that for $t > 0$ the angle between velocity and acceleration vectors is always acute

**Solution:**

$\dot{r} = c \ ,\ddot{r} = 0 \ , \quad \dot{\theta} = g \ , \quad \ddot{\theta} = 0$

$$\vec{v} = \dot{r}\boldsymbol{i_r} + (r\dot{\theta})\,\boldsymbol{j_\theta}$$

$$\therefore \vec{v} = c\boldsymbol{i_r} + (ct)g\,\boldsymbol{j_\theta}$$

$$\vec{v} = c(\boldsymbol{i_r} + gt\,\boldsymbol{j_\theta})$$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\boldsymbol{i_r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\boldsymbol{j_\theta}$$

$$\vec{a} = (0 - (ct)tg^2)\boldsymbol{i_r} + (0 + 2cg)\boldsymbol{j_\theta}$$

$$\vec{a} = -cg^2t\boldsymbol{i_r} + 2cg\boldsymbol{j_\theta} = cg(-gt\boldsymbol{i_r} + 2\boldsymbol{j_\theta})$$

$$\text{speed } |\vec{v}| = [c^2 + c^2g^2t^2]^{1/2} = [c(1^2 + g^2t^2)]^{1/2} = c(1 + g^2t^2)^{1/2}$$

To find the angle between $\vec{v}$ and $\vec{a}$

$$\vec{v}.\vec{a} = -c^2g^2t + 2c^2g^2t = c^2\, g(-gt + 2gt) = c^2g^2t$$

$\because \vec{v}.\ \vec{a} > 0$ for $t > 0$

Hence for $t > 0$ , the angle between $\vec{v}$ and $\vec{a}$ is acute

## 2.1 Newton's Laws of Motion

Newton's Laws of Motion are as follows:

1. Everybody continues in its ***state of rest*** or of ***uniform motion in a straight line***, unless it is compelled by a force to change that state.
2. The change of motion is ***proportional*** to the applied force and takes place in the direction of the force.
3. To every action, there is always an equal and opposite reaction or the mutual actions of two bodies are always equal and oppositely directed.

قوانين نيوتن بالحركة هي:

1. كل جسم يستمر على حالته من السكون أو الحركة المنتظمة على خط مستقيم ، مالم تؤثر عليه قوة لتغيير تلك الحالة.
2. يتناسب تغير الحركة مع القوة المسلطة ويكون باتجاه تاثيرالقوة.
3. لكل فعل، يكون هناك دائمًا رد فعل مساو له بالمقداروومعاكس بالاتجاه. الافعال المتبادلة لجسمين تكون دائما متساوية معاكسة بالاتجاه.

## 2.2 Newton's First Law: Inertial Reference Systems

The first law describes a common property of matter, namely, inertia.

***Inertia*** is the resistance of all matter to having its motion changed

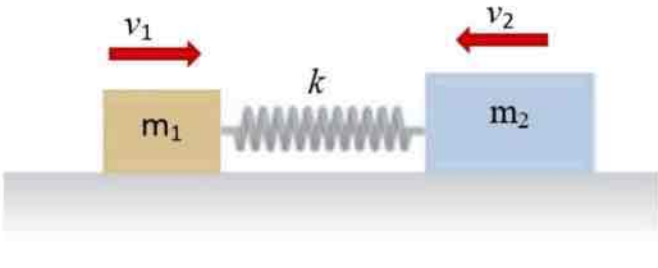
يصف القانون الأول خاصية مشتركة للمادة ، وهي القصور الذاتي.

القصور الذاتي هو مقاومة الجسم لتغير حركته

## 2.3 Mass and Force: Newton's Second and Third Law

Consider two masses $m_1$ and $m_2$ attached by a spring and they initially were at rest. If the two masses were pushed together, compressing the spring and then releasing them, so that they fly apart attaining speeds $\vec{v}_1$ and $\vec{v}_2$ respectively.

نفترض وجود كتلتين $m_1$ و $m_2$ مرتبطتين بنابض حلزوني وكانا في البداية في حالة سكون. إذا تم دفع الكتلتين باتجاه بعضهما ، عنها ينضغط النابض، ثم نتركهما بحيث ينفصلان عن بعضهما بسرع $\vec{v}_1$ و $\vec{v}_2$ على التوالي وباتجاهين متعاكسين.



The ratio of the two masses:

$$\frac{m_2}{m_1} = \left|\frac{\vec{v}_1}{\vec{v}_2}\right| \qquad \dots\dots(1)$$

Eqn. (1) is equivalent to:

$$\Delta(m_1\vec{v}_1) = -\Delta(m_2\vec{v}_2) \qquad \dots\dots(2)$$

because the ***initial*** velocities of each mass are ***zero*** and the final velocities $\vec{v}_1$ and $\vec{v}_2$ are in ***opposite*** directions. If we divide by $\Delta t$ and take limits as $\Delta t \to 0$ obtain:

$$\frac{d}{dt}(m_1\vec{v}_1) = -\frac{d}{dt}(m_2\vec{v}_2) \quad \dots\dots\dots.(3)$$

The product of mass and velocity, $m\vec{v}$, is called ***linear momentum.***

So the second law can be rephrased as follows: ***The time rate of change of an object's linear momentum is proportional to the impressed force***, ***F***. Thus, the second law can be written as:

يُطلق على حاصل ضرب الكتلة والسرعة الزخم الخطي.

لذلك يمكن إعادة صياغة القانون الثاني للحركة على النحو التالي: القوة تساوي المعدل الزمني لتغير الزخم الخطي للجسم .

$$\vec{F} \propto \frac{d}{dt}(m\vec{v})$$

$$\vec{F} = k\frac{d}{dt}(m\vec{v})$$

where k is a constant of proportionality. Let $k = 1$

$$\therefore \vec{F} = \frac{d}{dt}(m\vec{v})$$

where $m$ constant, finally express Newton's second law in the familiar form:

$$\vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a} \qquad \dots..(4)$$

$\vec{F}$: is the net force acting upon the mass $m$; that is, it is the vector sum of all of the individual forces acting upon $m$.

From Eqn. (3)

$$\vec{F}_1 = -\vec{F}_2 \qquad \dots\dots.(5) \qquad \textbf{\textit{Newton's third law}}$$

***Two interacting bodies exert equal and opposite force upon one another.***

كل جسمين يؤثر احدهما على الاخر بقوة متساوية بالمقدار ومتعاكسة بالاتجاه

## 2.4 Linear Momentum

$$\vec{P} = m\vec{v} \quad \dots\dots(1)$$

$$\vec{F} = \frac{d\vec{P}}{dt} \quad \dots\dots(2)$$

Sub Eqn. (2) in third Newton's law $\vec{F}_1 = -\vec{F}_2$

$$\frac{d\vec{P}_1}{dt} = -\frac{d\vec{P}_2}{dt}$$

$$\frac{d\vec{P}_1}{dt} + \frac{d\vec{P}_2}{dt} = 0 \quad or \quad \frac{d}{dt}(\vec{P}_1 + \vec{P}_2) = 0$$

$\therefore$ $\vec{P}_1 + \vec{P}_2 = \text{constant}$ (***conservation of linear momentum***)

$\therefore$ Newton's third law implies that the total momentum of two mutually interacting bodies is a constant.

The equation of motion for a particle subject to the influence of a net force, $\vec{F}$, writing as the vector sum of all the forces acting on the particle.

$$\vec{F} = \sum \vec{F}_i = m\frac{d^2\vec{r}}{dt^2} = m\vec{a}$$

كل جسمين يؤثر احدهما على الاخر بقوة متساوية ومعاكسة بالاتجاه

**Example:**

A spaceship of mass $M$ is traveling in deep space with initial velocity ($\vec{v}_i = 20\ km/s$) relative to the sun. It ejects a rear stage of mass ($0.2\ M$) with speed ( $\vec{u} = 5\ km/s$), find the final velocity $\vec{v}_f$ of the space ship after ejection.

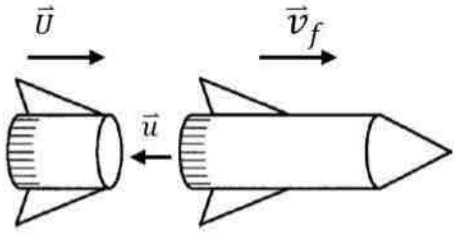
**Solution:**

The system of spaceship plus rear stage is a closed system upon which no external forces act; the total linear momentum is conserved.

نظام سفينة الفضاء اضافة الى المرحلة الخلفية هو نظام مغلق ليس عليه قوى خارجية ؛ عليه يكون الزخم الخطي الكلي محفوظ

$$\Delta\vec{P} = \vec{P}_f - \vec{P}_i = 0$$

$$\therefore \vec{P}_f = \vec{P}_i \dots\dots(1)$$




$\vec{P}_i$ = initial linear momentum

$\vec{P}_f$ = final linear momentum

Let U be the velocity of the rear stage after ejection.

$$\vec{P}_i = M\vec{v}_i \dots\dots(2)$$

The total momentum of the system after ejection is then:

$$\vec{P}_f = 0.2M\,\vec{U} + 0.8\,M\vec{v}_f \quad \dots\dots(3)$$

الزخم الخطي الكلي للنظام بعد الانفصال

$$\vec{u} = \vec{v}_f - \vec{\mathrm{U}}$$

$$\vec{\mathrm{U}} = \vec{v}_f - \vec{u} \qquad \dots\dots(4)$$

Sub. Eqn. (4) in Eqn. (3)

$$\vec{P}_f = 0.2M(\vec{v}_f - \vec{u}) + 0.8\,M\vec{v}_f \qquad \dots\dots(5)$$

Eqn. (5) equal Eqn.(2), so:

$$[0.2M(\vec{v}_f - \vec{u}) + 0.8\,M\vec{v}_f = M\,\vec{v}_i\,] \div M$$

$$0.2\vec{v}_f + 0.8\vec{v}_f = \vec{v}_i + 0.2\vec{u}$$

$$\vec{v}_f = \vec{v}_i + 0.2\,\vec{u}$$

$$= 20\,km/s + 0.2(5\,km/s)$$

$$\vec{v}_f = 21\,km/s$$

## 2.5 Rectilinear Motion

When a moving particle remains on a ***single straight line***, the motion is said to be "***rectilinear*** ". The general equation motion is:

$$\vec{F}\,(x, \dot{x}\,, t) = m\ddot{x} = m\vec{a}$$

عندما يستمر الجسم في حركته على خط مستقيم واحد، يُقال إن الحركة "مستقيمة"

**Note:** We usually use the single variable *x* to represent the position of a particle. To avoid unnecessary use of subscripts, we often use the symbols $v, a, \dot{x},\ \ddot{x}\ and\ F$ respectively, rather than $v_x, a_x, \dot{x}_x,\ \ddot{x}_x\ and\ F_x$.

**Special Cases:**

1. ***Constant Force***

$$\vec{F} = constant\ \ then\ \vec{a} = constant$$

$$\therefore \vec{F} = m\frac{d\vec{v}}{dt} \rightarrow \frac{d\vec{v}}{dt} = \frac{\vec{F}}{m} = \vec{a} = constant \quad \dots\dots(1)$$

$$m\ddot{x} = m\vec{a}$$

$$\vec{F} = \ddot{x} = \frac{d\vec{v}}{dt} = \vec{a}$$

$$d\vec{v} = \vec{a}dt$$

$\int_{u_0}^{u} d\vec{v} = \int_0^t \vec{a}\, dt$

$v - v_0 = at \qquad v_0 \equiv$ Initial velocity

$\vec{v} = at + v_0 \quad \ldots\ldots (2)$

$\therefore \vec{v} = \frac{dx}{dt}$

$\therefore \frac{dx}{dt} = at + v_0$

$\int_{x_0}^{x} dx = \int_0^t (at + v_0)\, dt$

$x - x_0 = \frac{1}{2}at^2 + v_0 t$

$x = \frac{1}{2}\, at^2 + v_0 t + x_0 \quad \ldots\ldots (3)$

$at = v - v_0$

$\therefore t = \frac{v - v_0}{a} \quad \ldots\ldots (4)$

$x - x_0 = \frac{1}{2}a\left(\frac{v-v_0}{a}\right)^2 + v_0\left(\frac{v-v_0}{a}\right)$

$2a\,(x - x_0) = a^2\left(\frac{v-v^2}{a}\right)^2 + 2v_0(v - v_0)$

$2a\,(x - x_0) = v^2 + v_0^2 - 2vv_0 + 2vv_0 - 2v_0^2$

$2a\,(x - x_0) = v^2 - v_0^2 \quad \ldots\ldots (5)$

The equations of ***uniformly accelerated motion***

$$\boxed{\begin{aligned} &\vec{v} = at + v_0 \\ &x = \frac{1}{2}\, at^2 + v_0 t + x_0 \\ &2a\,(x - x_0) = v^2 - v_0^2 \end{aligned}}$$

**2. *Free Fall***

In the case of a body ***falling freely*** near the surface of the Earth, neglecting air resistance, the acceleration is very nearly constant

في حالة السقوط الحر لجسم ما بالقرب من سطح الأرض، وإهمال مقاومة الهواء، يكون التعجيل ثابتًا

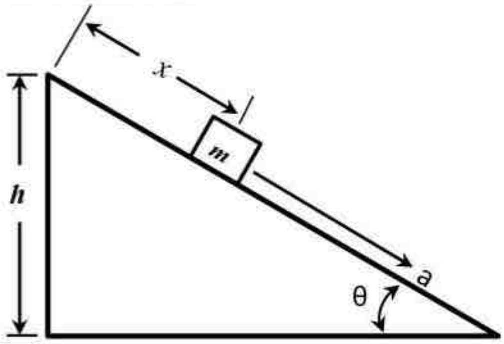
$a = g = 9.8\ \frac{m}{sec^2} = 32\ ft\,/sec^2$

$\vec{F} = m\vec{g}$

**Example:**

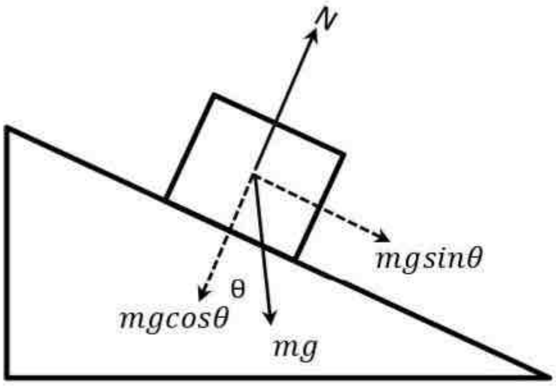
A block is sliding down on a smooth plane inclined at angle $\boldsymbol{\theta}$ to horizontal. If the height of the plane is $h$ as shown in the figure and the block is released from rest $(v_0 = 0)$ at the top, what will be its speed when it reaches the bottom? Then how the accelerate will become when surface is not smooth?

جسم ينزلق اسفل سطح املس يميل بزاوية $\theta$ عن الافق. اذا كان ارتفاع السطح هو $h$ كما هو موضح في الشكل وتم تحرير الجسم من السكون في الأعلى ، ما هي سرعته عندما يصل إلى اسفل السطح؟ وكم سيصبح تعجيله اذا كان السطح خشن




**Solution:**

**a)** ***Smooth plane*** **(*No Frictional force*)**




$\vec{F} = m\vec{a}$

$\vec{a} = \frac{\vec{F}}{m}$

$\therefore\ \vec{F} = mg\sin\theta$

$m\vec{a} = mg\sin\theta$

$\therefore \vec{a} = g\sin\theta \ldots\ldots (1)$

$x - x_0 = \frac{h}{\sin\theta} \ldots\ldots (2)$

Using one of the equation of motion $(2a\,(x - x_0) = (v^2 - v_0^2))$

where $v_0 = 0$

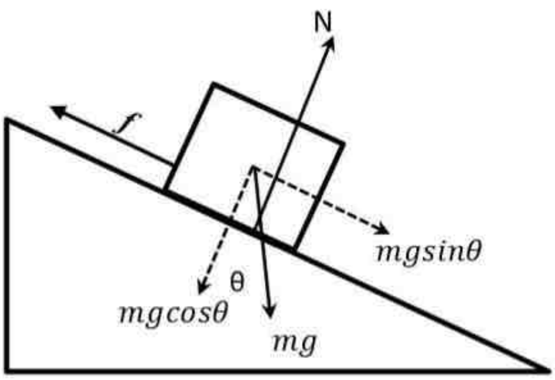
$\therefore v^2 = 2a(x - x_0) \ldots\ldots (3)$

$v^2 = 2(g\sin\theta)\left(\frac{h}{\sin\theta}\right)$

$\therefore v^2 = 2gh$

$v = \sqrt{2gh} \ \ldots\ldots (4)$

**b)** ***Rough Plane*** **(*Frictional force*)**




$\vec{F} = mg\sin\theta - f \ldots\ldots (5)$

$\vec{f}\,\alpha\,\vec{N}$

$\vec{f} = \mu\vec{N}$

$N$: normal force, $\mu$: coefficient of sliding or kinetic friction

From figure:

$N = mg\cos\theta$

$\therefore f = \mu mg\cos\theta \qquad \ldots\ldots(6)$

$\vec{F} = m\vec{a} = mg\sin\theta - \mu mg\cos\theta$

$\vec{a} = g\,(\sin\theta - \mu\cos\theta)$

For motion up the plane, the direction of the frictional force is reversed; that is, it is in the positive $x$ direction. The acceleration (actually ***deceleration***) is then:

$\vec{a} = g(\sin\theta + \mu\cos\theta)$

للحركة اعلى السطح، يتم عكس اتجاه قوة الاحتكاك. اي تكون في الاتجاه الموجب. التسارع (في الواقع هنا هو تباطؤ)

## 2.6 Forces that Depend on Position
## (The Concepts of Kinetic and Potential Energy)

مفهوم الطاقة الحركية والطاقة الكامنة

- Force depends only on the particles position
- Electrostatic and gravitational forces.
- Forces of elastic tension or compression.

- قوة تعتمد على موضع الجسيمات فقط
- القوى الكهروستاتيكية والجاذبية.
- قوى التوتر المرن أو الضغط.

If the force is *independent of velocity or time*, then the differential equation for rectilinear motion is simply:

$\vec{F}(x) = m\ddot{x} \qquad \ldots\ldots(1)$

Using the chain rule

$\therefore \ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dt}\frac{dx}{dx} = \frac{dx}{dt}\frac{d\dot{x}}{dx}$

$\therefore \ddot{x} = v\frac{dv}{dx} \qquad \ldots\ldots(2)$

Sub. Eqn. (2) in Eqn. (1)

$\therefore F(x) = mv\frac{dv}{dx} \qquad \ldots\ldots(3)$

Also, Eqn.(3) may be written as:

$\vec{F}(x) = \frac{m}{2}\frac{d(v^2)}{dx}$

$\vec{F}(x) = \frac{dT}{dx} \qquad \ldots\ldots(4)$

where $\left[T = \frac{1}{2}mv^2\right] \rightarrow$ Kinetic energy of the particle

$F(x)dx = dT$

$$\int_{x_0}^{x} F(x)dx = \int_{T_0}^{T} dT = T_x - T_0 = W \ldots\ldots (5)$$

The ***work*** is equal to the change in the kinetic energy of the particle.

Let us define a function $V(x)$ such that

الشغل يساوي التغير في الطاقة الحركية للجسم

$$F(x) = -\frac{dV}{dx}$$

$$\int_{x_0}^{x} F(x)\,dx = -V + constant \quad \ldots\ldots (6)$$

$V$: ***Potential energy***.

The function $V(x)$ is called the potential energy in terms of $(x)$, the work integral is

$$W = \int_{x_0}^{x} f(x)\,dx = -\int_{x_0}^{x} dV = -V(x) + V(x_0) = T - T_0$$

$$-V + costant = T$$

$$T + V = costant$$

$$\frac{1}{2}mv^2 + V(x) = costant = E \quad \ldots..(7)$$ ***Total energy equation***

Total energy (total mechanical energy) it is equal to the sum of the kinetic and potential energies and is constant throughout the motion of the particle.

Such force (depend on position only) called ***Conservative force***. ***Nonconservative forces*** that is, those for which no potential energy function exists are usually of a dissipational nature, such as friction.

(***Free Fall***) (Constant acceleration) is an example of ***conservative motion***.

إجمالي الطاقة يساوي مجموع الطاقات الحركية والاحتمالية وهو ثابت طوال حركة الجسيم. ان هذه القوة (تعتمد على الموضع فقط) والتي تسمى القوة المحافظة. عادة ما تكون القوى غير المحافظة ، تلك التي لا توجد لها دالة طاقة كامنة ، فتكون اعتياديا من نوع التبديد، مثل الاحتكاك. في حين ان السقوط الحر (تعجيل ثابت) هو مثال على الحركة المحافظة.

$$\frac{1}{2}mv^2 + V(x) = E$$

$$\frac{1}{2}mv^2 = E - V(x) \qquad *\frac{2}{m}$$

$$v^2 = \frac{2}{m}[E - V(x)]$$

$$\therefore \vec{v} = \mp\sqrt{\frac{2}{m}[E - V(x)]} \quad \ldots\ldots(8)$$ ***Equation of velocity as a function of (x)***

$$\therefore \vec{v} = \frac{dx}{dt} = \mp\sqrt{\frac{2}{m}[E - V(x)]}$$

$$dt = \mp\frac{dx}{\sqrt{\frac{2}{m}[E - V(x)]}}$$

$t = \int_{x_0}^{x} \frac{\mp\, dx}{\sqrt{\frac{2}{m}[E - V(x)]}}$ ……(9) ***Equation of time (t) as a function of (x)***

**Note that:**

1. When $V(x) \le E \rightarrow The\ \boldsymbol{velocity}\ (v)\ is\ \boldsymbol{real}$
2. When $V\ (x) = E\ \ \rightarrow The\ \boldsymbol{velocity}\ (v) = 0$

   This means that the particle must come to rest and reverse its motion at points for which the equality holds. These points are called the ***turning points*** of the motion.
3. When $V(x) \ge E\ \ \rightarrow The\ \boldsymbol{velocity}\ (v)$ is $\boldsymbol{imaginary}$

- تكون السرعة $(v)$ حقيقية في حالة كون $V(x)$ اقل او مساوية للطاقة الكلية للجسم.
- تكون السرعة مساوية للصفر في حالة كون $V(x)$ مساوية للطاقة الكلية وهذا يعني أن الجسيم يجب أن يقف ويعكس حركته عند نقاط تحمل المساواة فيها. وتسمى هذه النقاط تسمى نقاط الرجوع للحركة.
- تكون السرعة ذات قيمة خيالية في حالة كون $V(x)$ مساوية للطاقة الكلية للجسم.

**Example: (*Free Fall*)**

A body is projected upward in the positive *x*-direction with initial speed $(v_0)$. choosing $x = 0$ as initial point of projection , find the maximum height attained by the body and then find the equation of time $(t)$ in terms of $(g)$

**Solution:**

Choose the *x* direction to be positive upward, and then the gravitational force is equal to $(-mg)$.

At $x = 0$ $\quad \dot{x} = v_0$

At $x = max$ $\quad \dot{x} = ?$

$F = -mg \ldots\ldots(1)$

$F = -\frac{dV(x)}{dx}$

$\therefore \int F(x)dx = -V + c \qquad \ldots\ldots(2)$

Sub Eqn. (1) in Eqn. (2)

$\therefore \int -mg\ dx = -V + c$

$-mgx = -V + c$

choose $c = 0$

then $V = 0$ at $x = 0$

$\therefore V = mgx \ldots\ldots(3)$ $\qquad$ $\boldsymbol{Potential\ energy}$

$E = \frac{1}{2}\ mv^2 + V \ldots\ldots(4)$

Sub Eqn. (3) in Eqn. (4)

$$\therefore E = \frac{1}{2}mv^2 + mgx \quad \dots\dots(5)$$

The body be projected upward with ***initial speed*** $\boldsymbol{v_0}$ from the ***origin*** $\boldsymbol{x = 0}$

$$E = \frac{1}{2}mv_0^2 + mg(0)$$

$$\therefore E = \frac{1}{2}\, mv_0^2 \quad \dots\dots(6)$$

Energy equation during body motion

$$\therefore \frac{1}{2}\, mv_0^2 = \frac{1}{2}mv^2 + mgx \quad \dots\dots(7)$$

$$v^2 = v_0^2 - 2gx \quad \dots\dots(8)$$

The ***turning point*** of the motion, which is in this case the maximum height $(X = X_{max})$ , is given by setting $v = 0$.

نقطة الرجوع في الحركة، تكون في هذه الحالة عند أقصى ارتفاع

$$0 = v_0^2 - 2gX_{max}$$

$$\frac{1}{2}\, m\, v_0^2 = mgX_{max}$$

$$\therefore \boxed{X_{max} = \frac{v_0^2}{2g}}$$ →The maximum height that the body attained

To obtain ***time*** $\boldsymbol{(t)}$ from Eqn. (8)

$$v^2 = v_0^2 - 2gx$$

$$v^2 = \left(\frac{dx}{dt}\right)^2 = v_0^2 - 2gx$$

$$\frac{dx^2}{dt^2} = v_0^2 - 2gx$$

$$dt^2 = \frac{dx^2}{v_0^2 - 2gx}$$

$$dt^2 = (v_0^2 - 2gx)^{-1} dx^2$$

$$dt = (v_0^2 - 2gx)^{-1/2} dx$$

By integration two side:

$$\int_0^t dt = \int_0^x (v_0^2 - 2gx)^{-1/2}\, dx$$

$$t = -\left.\frac{2(v_0^2 - 2gx)^{1/2}}{2g}\right|_0^x = -\frac{(v_0^2 - 2gx)^{1/2}}{g} + \frac{v_0}{g}$$

$$t = \frac{v_0}{g} - \frac{(v_0^2 - 2gx)^{1/2}}{g}$$

## 2.7 Variation of Gravity with Height

We assumed that $g$ was constant. Actually, the force of gravity between two particles is inversely proportional to the square of the distance between them (***Newton's law of gravity***). The ***gravitational force*** that the ***Earth exerts on a body of mass m*** is given by:

افترضنا أن $g$ ثابت في الواقع، إن قوة الجاذبية بين جسمين تتناسب عكسيا مع مربع المسافة بينهما (قانون الجذب العام لنيوتن).

$$F_r = -G\frac{Mm}{r^2} \quad .......(1)$$

Where $G$: is Newton's constant of gravitation

$M$: is the mass of the Earth

$r$: is the distance from the center of the Earth to the body.

We know that there is a relation between the force and potential energy:

$$F = -\frac{\partial V}{\partial r} \rightarrow \partial V = -F\partial r \rightarrow dV = -Fdr \rightarrow F$$

$$\int dV = -\int -\frac{GMm}{r^2}dr$$

$$V(r) = GMm\left(\frac{r^{-2+1}}{-1}\right)$$

$$V(r) = -\frac{GMm}{r} \quad .........(2)$$ ***Potential Energy function***

If we ***neglect air resistance,*** the differential equation of motion is:

في حالة اهمال مقاومة الهواء

$$m\ddot{r} = -G\frac{Mm}{r^2} \quad ......(3)$$

$$\ddot{r} = \frac{d\dot{r}}{dt} \quad * \frac{dr}{dr}$$

$$\ddot{r} = \frac{d\dot{r}}{dt}\frac{dr}{dr} = \frac{d\dot{r}}{dr}\frac{dr}{dt}$$

$$\therefore \ddot{r} = \dot{r}\frac{d\dot{r}}{dr} \quad ......(4)$$

Sub. Eqn. (4) in Eqn.(3)

$$m\dot{r}\frac{d\dot{r}}{dr} = -G\frac{Mm}{r^2}$$

Integrating both side with respect $\dot{r}$ and $r$

$$m\int \dot{r}d\dot{r} = -GMm\int\frac{dr}{r^2}$$

$$m\int \dot{r}d\dot{r} = -GMm\int r^{-2}\,dr$$

$$\frac{1}{2}m\dot{r}^2 = -\frac{GMm}{-1}\,r^{-2+1} = GMmr^{-1} + c$$
$$\frac{1}{2}m\dot{r}^2 = \frac{GMm}{r} + c$$
$$\frac{1}{2}m\dot{r}^2 = \frac{GMm}{r} + c \quad \dots\dots(5)$$

$c = E$ is constant of integration.

$$\therefore \boxed{\frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} = E} \quad \dots\dots(6)$$ ***Free Falling Energy equation***

Eqn. (6) is ***energy equation***, represent the sum of the ***kinetic energy*** ($1^{st}$ term) and the ***potential energy*** ($2^{nd}$ term) remain constant throughout the motion of a ***falling body***.

المعادلة رقم (6) تمثل معادلة الطاقة، وتمثل مجموع الطاقة الحركية (الحد الأول) والطاقة الكامنة (الحد الثاني) والتي تبقى ثابتة طوال حركة الجسم الساقط.

When the projectile shot upward from the surface of the earth with initial speed $v_0$:

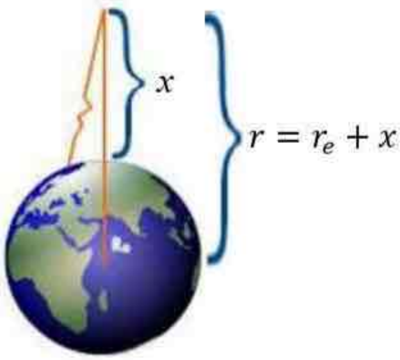
$$\frac{1}{2}mv_0^2 - \frac{GMm}{r_e} = E \quad \dots\dots(7)$$

Where $r_e$ is the ***radius of the earth***

Now, in order to find the speed of the projectile at any height $x$ above the earth's surface, combining the last two energy equations (6) and (7):

الآن، لايجاد سرعة القذيفة في أي ارتفاع $x$ فوق سطح الأرض، نجمع المعادلتين الأخيرتين للطاقة (6) و (7)

$$\frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} = \frac{1}{2}mv_0^2 - \frac{GMm}{r_e}$$
$$\frac{1}{2}mv_0^2 - \frac{1}{2}m\dot{r}^2 + \frac{GMm}{r} - \frac{GMm}{r_e} = 0$$
$$\frac{1}{2}m(v_0^2 - \dot{r}^2) + GMm\left(\frac{1}{r} - \frac{1}{r_e}\right) = 0$$




Substituting by ($v = \dot{r}$), then multiply the Eqn. by $\left(\frac{2}{m}\right)$, we get:

$$(v_0^2 - v^2) + 2GM\left(\frac{1}{r} - \frac{1}{r_e}\right) = 0$$
$$v^2 = v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{r_e}\right)$$

But $r = r_e + x$

$\therefore v^2 = v_0^2 + 2GM\left(\frac{1}{r_e+x} - \frac{1}{r_e}\right)$ ..... (8) ***Speed at any height above the earth surface***

Now the equation of gravity acceleration for the projectile on the earth's surface is given: (*gravitational force equal to weight of body*)

الآن يتم إعطاء معادلة تعجيل الجاذبية للقذيفة على سطح الأرض: (قوة الجاذبية تساوي وزن الجسم)

$$-\frac{GMm}{r_e^2} = -mg$$

$$g = \frac{GM}{r_e^2} \quad \ldots\ldots (9)$$

$$G = \frac{gr_e^2}{M} \quad \ldots\ldots (10)$$

Sub. Eqn. (10) in Eqn. (8)

$$v^2 = v_0^2 + \frac{2Mgr_e^2}{M}\left(\frac{1}{r_e+x} - \frac{1}{r_e}\right)$$

$$v^2 = v_0^2 + 2gr_e^2\left(\frac{r_e-(r_e+x)}{r_e(r_e+x)}\right)$$

$$v^2 = v_0^2 + 2gr_e^2\left(\frac{r_e-r_e-x}{r_e(r_e+x)}\right)$$

$$v^2 = v_0^2 + 2gr_e^2\left(\frac{-x}{r_e(r_e+x)}\right)$$

$$v^2 = v_0^2 - 2gx\left(\frac{r_e^2}{r_e(r_e+x)}\right)$$

$$v^2 = v_0^2 - 2gx\left(\frac{r_e}{r_e+x}\right)$$

$$v^2 = v_0^2 - 2gx\left[\frac{\frac{r_e}{r_e}}{\left(\frac{r_e}{r_e}+\frac{x}{r_e}\right)}\right] = v_0^2 - 2gx\left[\frac{1}{1+\frac{x}{r_e}}\right]$$

$v^2 = v_0^2 - 2gx(1+\frac{x}{r_e})^{-1}$.....(11) ***Speed of projectile with variant gravity acceleration***

When $x \ll re \rightarrow \left(\frac{x}{r_e}\right)$ can be neglected, then Eqn. (11) reduces to the form:

$v^2 = v_0^2 - 2gx$ ......(12) ***Speed of projectile with uniform gravitational field***

The ***maximum height*** (***turning point***) is found by setting $v = 0$ and solving for $x$

$$\therefore 0 = v_0^2 - 2gx_{max}(1+\frac{x_{max}}{r_e})^{-1}$$

$$v_0^2 = 2gx_{max}(1+\frac{x_{max}}{r_e})^{-1}$$

$x_{max} = \frac{v_0^2}{2g}(1 + \frac{x_{max}}{r_e})$

$x_{max} = \frac{v_0^2}{2g} + \frac{v_0^2}{2g}\frac{x_{max}}{r_e}$

$x_{max} - \frac{v_0^2}{2g}\frac{x_{max}}{r_e} = \frac{v_0^2}{2g}$

$x_{max}(1 - \frac{v_0^2}{2gr_e}) = \frac{v_0^2}{2g}$

$x_{max} = \frac{v_0^2}{2g}\left(1 - \frac{v_0^2}{2gr_e}\right)^{-1}$ .....(13) ***Maximum height of the projectile***

Again, if $v_0^2 \ll 2gr_e \rightarrow \frac{v_0^2}{2gr_e}$ *can be neglected*

$x_{max} = h = \frac{v_0^2}{2g}$ ........(14) ***Maximum height of the projectile with low initial speed***

To find $v_0$ that make the projectile escape from the earth's gravity, which is called ***escape speed***, we need to expand the series in Eqn. (13), by using binomial, as in:

لايجاد $v_0$ التي تجعل الجسم المقذوف يهرب من جاذبية الأرض، والتي تسمى سرعة الهروب، نحتاج إلى ايجاد مفكوك المتسلسلة في معادلة (13)، باستخدام متعدد الحدود.

$(1 - \frac{v_0^2}{2gr_e})^{-1} = \left(1 - \frac{v_0^2}{2gr_e} + \left(\frac{v_0^2}{2gr_e}\right)^2 - \cdots\right)$

Sub. in Eqn. (13)

$x_{max} = h = \frac{v_0^2}{2g}\left(1 - \frac{v_0^2}{2gr_e} + \left(\frac{v_0^2}{2gr_e}\right)^2 - \cdots\right)$

$x_{max} = h = \frac{v_0^2}{2g} - \left(\frac{v_0^2}{2g}\right)^2\frac{1}{r_e} + \left(\frac{v_0^2}{2g}\right)^3\frac{1}{r_e^2} + \cdots$

Neglecting high terms

$x_{max} = h = \frac{v_0^2}{2g}$

$v_0^2 = 2gh$

$v_0 = \sqrt{2gh}$

$h = x_{max} = r_e = 6.4 \times 10^6 m$ and $g = 9.8\,\mathrm{m}/s^2$

$\therefore v_e = \sqrt{2g\, r_e} \cong 11 km/s$ ......(15) ***Escape velocity***

In the ***Earth's atmosphere***, the average speed of air molecules ($O_2$ and $N_2$) is about 0.5 *km/s*, which is considerably less than the escape speed, so the Earth retains its atmosphere. The ***moon***, has no atmosphere; because the escape speed at the moon's surface, owing to the moon's small mass, is considerably smaller than that at the Earth's surface, any oxygen or nitrogen would eventually disappear.

في الغلاف الجوي للأرض ، يبلغ متوسط سرعة جزيئات الهواء ($O_2$ و $N_2$) حوالي 0.5 كم / ثانية ، وهو أقل بكثير من سرعة الهروب، لذلك تحتفظ الأرض بجوها. القمر ليس له جو، لأن سرعة الهروب عند سطح القمر، بسبب كتلة القمر الصغيرة، أصغر بكثير من سرعة الهروب عند سطح الأرض، فإن أي أوكسجين أو نيتروجين سيختفي في النهاية.

## 2.8 The Force as a Function of Velocity Only
### (*Horizontal Motion with Liner Resistance*)

It often happens that the force that acts on a body is a *function of the velocity of the body*. This is true, for example, in the case of viscous resistance exerted on a body moving through a fluid. If the force can be expressed as a function of *v* only, the differential equation of motion may be written in either of the two forms

غالبا ما يحدث أن القوة المؤثرة على الجسم هي دالة لسرعة الجسم. هذا صحيح، على سبيل المثال، في حالة مقاومة اللزوجة التي تؤثرعلى الجسم المتحرك عبر مائع. إذا أمكن التعبير عن القوة كدالة للسرعة فقط ، فيمكن كتابة المعادلة التفاضلية للحركة في صيغيتين

$$F_0 + F(v) = m\frac{dv}{dt} \ldots\ldots (1)$$

$F_0 =$ is constant force that $\boldsymbol{dose\ not\ depend\ on\ v}$

$$mdv = F(v)dt$$

$$dt = \frac{m\,dv}{F(v)}$$

By integrating both sides:

$$dt = \int \frac{m\,dv}{F(v)} \Longrightarrow t \rightarrow t(v) \ldots\ldots (2)$$

Assuming that we can solve the above Eqn. for $(v)$

$$(v) = v(t)$$

Second integration: $\int v(t)dt = x(t)$

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

Sub. in Eqn. (1)

$$F_0 + F(v) = mv\frac{dv}{dx}$$

$$dx = \frac{mv\,dv}{F(v)}$$

$$x = \int \frac{mv\,dv}{F(v)} = x(v) \quad \ldots\ldots(3)$$ ***Position as function of*** ($v$)

**Example: (*Horizontal Motion with Liner Resistance)***

A block is projected with initial velocity $(v_0)$ on a smooth horizontal plane, but it was affected by air resistance proportional of ($v$) i.e. $F(v) = -cv$. Find the equation of time $(t)$ as a function of $(v)$, then find the equation of velocity and displacement as a function of ($t$).

**Solution:**

$$F = -cv \ldots\ldots(1)$$

$$F = m\frac{dv}{dt} \ldots\ldots(2)$$

$$-cv = m\frac{dv}{dt}$$

$$dt = -\frac{m}{c}\frac{dv}{v}$$

By integrating both sides

$$\int_0^t dt = -\frac{m}{c}\int_{v_0}^{v}\frac{dv}{v}$$

$$t = -\frac{m}{c}\ ln\, v]_{v_0}^{v} = -\frac{m}{c}(\ln v - \ln v_0)$$

$$t = -\frac{m}{c}\ln\left(\frac{v}{v_0}\right) \quad \ldots\ldots..(3)$$ ***Equation of time as a function of*** $(\boldsymbol{v})$

Multiplying by $\left(\frac{-c}{m}\right)$

$$\frac{-c\,t}{m} = \ln\frac{v}{v_0}$$

$$e^{\frac{-c\,t}{m}} = \frac{v}{v_0}$$

$$v = v_0\ e^{\frac{-c\,t}{m}} \quad \ldots\ldots(4)$$ ***Velocity as a function of*** $(\boldsymbol{t})$

$$v = \frac{dx}{dt}$$

$$\therefore \frac{dx}{dt} = v_0\, e^{\frac{-c\,t}{m}}$$

$$\int_0^x dx = \int_0^x v_0\, e^{\frac{-c\,t}{m}}\, dt \ldots\ldots (5)$$

$$x = \int_0^t v_0\, e^{\frac{-c\,t}{m}}\ dt * \frac{-\frac{c}{m}}{-\frac{c}{m}}$$

$$x = -\frac{m\,v_0}{c}\int_0^t \frac{-c}{m}\, e^{\frac{-c\,t}{m}} dt$$

$$x = -\frac{m\,v_0}{c}\, e^{\frac{-c\,t}{m}}\Big]_0^t$$

$$x = -\frac{m\,v_0}{c}\, e^{\frac{-c\,t}{m}} + \frac{m\,v_0}{c}\, e^{\frac{-c(o)}{m}}$$

$$x = \frac{m\,v_0}{c}\left(1 - e^{\frac{-c\,t}{m}}\right) \quad \ldots\ldots.(6)$$ ***Displacement as a function of*** $(\boldsymbol{t})$

**Example:**

If $F = -cv$ find the velocity and time equations as a function of displacement for a particle with initial velocity $v_0$ .

**Solution:**

$$F = -cv \qquad \text{and} \qquad F = mv\frac{dv}{dx}$$

$$-cv = mv\frac{dv}{dx}$$

$$-\frac{c}{m}\int_0^x dx = \int_{v_0}^v dv$$

$$-\frac{c}{m}x = v - v_0$$

$$v = v_0 - \frac{c}{m}x \quad \ldots..(7)$$ ***Velocity as a function of*** $(\boldsymbol{x})$

[The speed of the body varies linearly with the displacement (distance)]

سرعة الجسم تتغير خطياً مع الازاحة

$$v = v_0 - \frac{c}{m}x$$

$$\therefore dv = -\frac{c}{m}dx$$

$$\int \frac{dv}{v} = \ln v$$

$$\therefore \int \frac{dv}{v} = \int \frac{-\frac{c}{m}dx}{v_0 - \frac{c}{m}x} = \ln\left[v_0 - \left(\frac{c}{m}\right)x\right]$$

$$\therefore v = \frac{dx}{dt}$$

$$\therefore \frac{dx}{dt} = v_0 - \frac{c}{m}x$$

$$\int_0^t dt = \int_0^x \frac{dx}{v_0 - \frac{c}{m}x} * \frac{-\frac{c}{m}}{-\frac{c}{m}}$$

$$t = \int_0^x \frac{-\frac{c}{m}dx}{-\frac{c}{m}[v_0 - \left(\frac{c}{m}\right)x]}$$

$$t = -\frac{m}{c}\ln\left(v_0 - \left(\frac{c}{m}\right)x\right)\Big]_0^x$$

$$t = \frac{-m}{c}\left[\ln\left(v_0 - \left(\frac{c}{m}\right)x\right) - \ln v_0\right]$$

$$t = -\frac{m}{c}\ln\left[\frac{v_0 - \left(\frac{c}{m}\right)x}{v_0}\right] \qquad \text{(time as a function of } (x))$$

## 2.9 The Force as a Function of Time Only

$$F(t) = m\frac{dv}{dt}$$

$$dv = \frac{F(t)}{m}dt$$

By integrating

$$v(t) = \int \frac{F(t)}{m}\, dt \;\ldots\ldots (1)$$

$$v(t) = \frac{dx}{dt}$$

$$dx = v(t)dt$$

$$\int dx = \int v(t)\, dt \quad \ldots\ldots. (2)$$

Sub Eqn. (1) in (2)

$$x = \int[\int \frac{F(t)}{m} dt]\, dt \ldots\ldots (3)$$

**Example:**

A block is initially at rest on a smooth horizontal surface. At time ($t = 0$) a constant increasing horizontal force is applied $F = ct$. Find the velocity and displacement as a function of time.

**Solution:**

$$F = ct = m\frac{dv}{dt}$$

$$dv = \frac{1}{m}ct\; dt$$

$$v = \frac{1}{m}\int_0^t ct\; dt$$

$$v = \frac{ct^2}{2m}$$

$$\therefore v = \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{ct^2}{2m}$$

$$dx = \frac{ct^2}{2m} dt$$

$$x = \int_0^t \frac{ct^2}{2m} dt$$

$$x = \frac{ct^3}{6m}$$

## 2.10 Vertical Motion in a Resisting Medium Terminal Velocity

### *Linear Resistance*

An object falling vertically through the air or through any fluid is subject to ***viscous resistance***. If the resistance is ***proportional*** to the ***first power*** of $(v)$, we can express this force as $(-c\vec{v})$ regardless of the sign of $(v)$ because the resistance is always ***opposite*** to the direction of motion. The constant of proportionality c depends on the ***size and shape of the object*** and the ***viscosity of the fluid***.

يتعرض الجسم الذي يسقط رأسياً عبر الهواء أو من خلال أي مائع لمقاومة اللزوجة. إذا كانت المقاومة تتناسب مع القوة الأولى لـ $(v)$ (الحالة الخطية)، فيمكننا التعبير عن هذه القوة كـ $(-c\vec{v})$ بغض النظر عن اشارة $(v)$لأن المقاومة تكون دائمًا معاكسة لاتجاه الحركة. ثابت التناسب c يعتمد على حجم وشكل الجسم ولزوجة المائع.

Let us take the $x$ axis to be ***positive*** upward. The differential equation of motion is then

$$-mg - c\vec{v} = m\frac{d\vec{v}}{dt} \quad \ldots\ldots(1) \qquad \textbf{\textit{Linear Equation}}$$

$$\int_0^t dt = \int_{v_0}^{v} \frac{mdv}{-mg-cv}$$

$$\frac{t}{m} = \int_{v_0}^{v} \frac{dv}{-mg-cv} = -\int_{v_0}^{v} \frac{dv}{(mg+cv)}$$

$$\frac{t}{m} = -\int_{v_0}^{v} \frac{dv}{mg+cv}$$

$$\frac{t}{m} = -\frac{1}{c_1} \ln(mg + cv)\Big]_{v_0}^{v}$$

$$\frac{t}{m} = -\frac{1}{c_1}[\ln(mg + cv) - \ln(mg + v_0)]$$

$$\therefore t = -\frac{m}{c}\ln\left(\frac{mg+cv}{mg+cv_0}\right) \quad \ldots\ldots(2) \qquad \textit{Equation of Time}$$

Eqn. (2) represents time in term of velocity; by solve Eqn. (2)

$$e^{\frac{-ct}{m}} = \frac{mg+cv}{mg+cv_0}$$

$$mg + cv = (mg + cv_0)e^{\frac{-ct}{m}}$$

$$cv = -mg + mg\, e^{\frac{-ct}{m}} + cv_0\, e^{\frac{-ct}{m}} \quad ] \div c$$

$$\therefore v = -\frac{mg}{c} + \frac{mg}{c}\, e^{\frac{-ct}{m}} + v_0\, e^{\frac{-ct}{m}}$$

$$\therefore v = -\frac{mg}{c} + \left(\frac{mg}{c} + v_0\right) e^{\frac{-ct}{m}} \ldots\ldots(3) \qquad \textit{Velocity as function of time}$$

Eqn. (3) represents velocity in term of time.

When $\left(t \gg \frac{m}{c}\right)$ *then* $e^{\frac{-ct}{m}} = 0$ so, the exponential term can be neglected

$$\therefore v = -\frac{mg}{c} \quad \ldots\ldots(4) \qquad \textbf{Terminal velocity}$$

***Terminal velocity*:** It is that velocity at which the force resistance is just equal and opposite to the weight of the body so that the total force on the body is zero and so the acceleration is zero.

سرعة المنتهى: هي تلك السرعة التي تكون فيها مقاومة القوة مساوية تماما ومعاكسة لوزن الجسم بحيث تكون القوة الكلية على الجسم تساوي الصفر وبالتالي فإن التعجيل سيكون صفر.

$\frac{mg}{c} = v_t$ ***Terminal speed*** سرعة المنتهى

$\tau = \frac{m}{c}$ ***Characteristic time*** الزمن النوعي

Then Eqn. (3) becomes:

$v = -v_t + (v_t + v_0)e^{-t/\tau}$ ……(4) ***Velocity in term of terminal velocity***

These two terms represent two velocities; the terminal velocity $v_t$ which exponentially (fades in) and the initial velocity $v_0$ which exponentially (fades out) due to the action of the viscous drag force.

هذان الحدان يمثلان سرعتين. سرعة المنتهى $v_t$ التي تتلاشى أسياً والسرعة الأبتدائية $v_0$ التي تتلاشى أسياً بسبب تأثير قوة اللزوجة.

We can find displacement by integrate Eqn. (3)

$$\therefore \frac{dx}{dt} = -\frac{mg}{c} + \left(\frac{mg}{c} + v_0\right) e^{\frac{-ct}{m}}$$

$$\int_{x_0}^{x} dx = -\frac{mg}{c}\int_0^t dt + \frac{mg}{c}\int_0^t e^{\frac{-ct}{m}}\,dt + \int_0^t v_0\, e^{\frac{-ct}{m}}\,dt$$

$$x - x_0 = -\frac{mg}{c}t + \frac{mg}{c}\left(\frac{-c/m}{-c/m}\right)\int_0^t e^{\frac{-ct}{m}}dt + \frac{v_0(-c/m)}{(-c/m)}\int_0^t e^{\frac{-ct}{m}}\,dt$$

$$x - x_0 = -\frac{mg}{c}t - \frac{m^2 g}{c^2}\int_0^t \frac{-c}{m} e^{\frac{-ct}{m}}dt \int_0^t -\frac{v_0 m}{c}\frac{-c}{m} e^{\frac{-ct}{m}}dt$$

$$x - x_0 = -\frac{mg}{c}t - \frac{m^2 g}{c^2}\left[e^{\frac{-ct}{m}}\right]_0^t - \frac{v_0 m}{c}\left[e^{\frac{-ct}{m}}\right]_0^t$$

$$x - x_0 = -\frac{mg}{c}t - \frac{m^2 g}{c^2}\left[e^{\frac{-ct}{m}} - e^0\right] - \frac{v_0 m}{c}\left[e^{\frac{-ct}{m}} - e^0\right]$$

$$x - x_0 = -\frac{mg}{c}t - \frac{m^2 g}{c^2}\left(e^{\frac{-ct}{m}} - 1\right) - \frac{v_0 m}{c}\left(e^{\frac{-ct}{m}} - 1\right)$$

$$x - x_0 = -\frac{mg}{c}t + \frac{m^2 g}{c^2}\left(1 - e^{\frac{-ct}{m}}\right) + \frac{v_0 m}{c}\left(1 - e^{\frac{-ct}{m}}\right)$$

$$x - x_0 = -\frac{mg}{c}t + \left(\frac{m^2 g}{c^2} + \frac{mv_0}{c}\right)\left(1 - e^{\frac{-ct}{m}}\right) \quad \ldots\ldots(5)$$ ***Displacement Equation***

Also, write in the form

$$x = x_0 - v_t t + X_1\left(1 - e^{\frac{-ct}{m}}\right) \quad \ldots\ldots(6)$$

where $X_1 = \frac{m^2 g}{c^2} + \frac{mv_0}{c} = g\tau^2 + v_0\tau$

***In particular***, for an object dropped from ***rest*** $v_0 = 0$ ,

From Eqn. (4)

$$v = -v_t + (v_t + v_0)e^{-t/\tau} = -v_t + v_t\, e^{-t/\tau}$$

- When $(t = \tau)$

  $v = \left(1 - e^{-t/\tau}\right)v_t$

  $v = (1 - e^{-1})v_t$

$\therefore When\ (t = \tau) \rightarrow v = (1 - e^{-1})vt$

- When $t = 2\tau$

  $v = \left(1 - e^{-2\tau/\tau}\right)v_t$

  $v = (1 - e^{-2})v_t$

$\therefore when\ (t = 2\tau) \rightarrow v = (1 - e^{-2})v_t$

Thus, after *one characteristic time* the speed is $(1 - e^{-1})$ *times* the terminal speed, after *two characteristic times* it is the factor $(1 - e^{-2})$ of and so on. After an interval of the speed is within 1% of the terminal value, namely, $(1 - e^{-5})v_t = 0.99995\ v_t$

وبالتالي اذا اسقط جسم من السكون فبعد زمن نوعي واحد، تكون السرعة $(1 - e^{-1})$ أضعاف سرعة المنتهى ، وبعد ضعفين للزمن النوعي تكون سرعته $(1 - e^{-2})$هكذا. بعد فترة زمنية تصل سرعة الجسم في حدود 1 ٪ من قيمة سرعة المنتهى ، أي $(1 - e^{-5})v_t = 0.99995\ v_t$

## 3.1 Introduction

Everywhere around us we see systems engaged in a ***periodic motion***: the small *oscillations* of a pendulum clock, a child playing on a *swing*, the rise and fall of the *tides*, the *swaying* of a tree in the wind and the *vibrations* of the strings on a violin. The essential feature that all these phenomena have in common is ***periodicity***, a pattern of movement or displacement that repeats itself over and over again.

في كل مكان حولنا نرى أنظمة تعمل بحركة دورية: التذبذبات الصغيرة لساعة البندول و تأرجح الطفل وهو يلعب على أرجوحة وصعود وانحدار المد والجزر وتمايل الاشجار في الريح واهتزازات أوتار الكمان. الميزة الأساسية التي تشترك فيها كل هذه الظواهر هي الدورية، والتي هي نمط من الحركة أو الازاحة الذي يعيد نفسه مرارًا وتكرارًا.

### 3.2 Linear Restoring Force, Harmonic Motion

One of the most important cases of rectilinear motion is that produced by a ***linear restoring force***. This force whose magnitude is **proportional** to the **displacement** of the particle from the **equilibrium position** and whose **direction** is always **opposite** to that of the displacement. Such a force is exerted by a spring obeying Hooke's law.

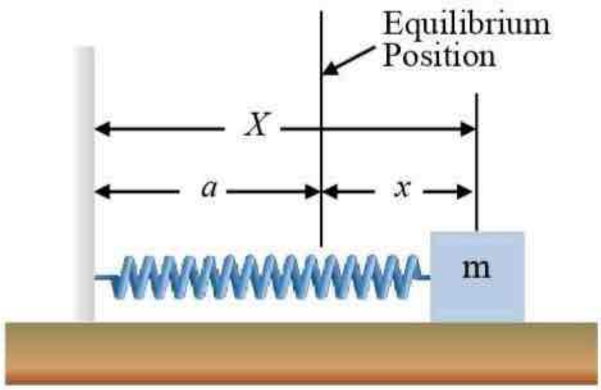
واحدة من أهم حالات الحركة على خط مستقيم تلك التي تحدثها قوة معيدة خطية. هذه القوة التي يتناسب مقدارها مع إزاحة الجسم من موضع التوازن اتجاهها يكون دائماً عكس اتجاه الازاحة. قوة كهذه يسببها وتر أو نابض يخضع لقانون هوك.

$x = X - a \;\; \dots\dots (1)$

Eq. (1) represents the displacement of the spring from its equilibrium length.

$X$ is total length

$a$ is upstretched (zero) length of the spring




Elastic force exerted by a spring obeying ***Hooke's law***

$F \, \alpha - x$

$F = -kx$ ……(2) ***Restoring Force (Hooke's law)***

where $k$ is called ***Stiffness (Spring Constant)*** معامل المرونة

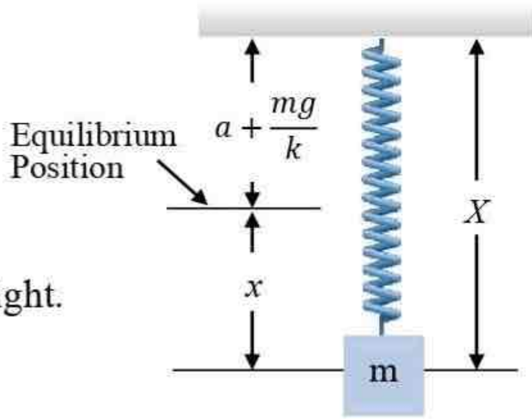
Sub. Eq.(1) in Eq.(2)

$\therefore F = -k(X - a) \dots\dots (3)$

Let the same spring be held vertically. The total force acting on the particle is:

$F = -kx$

$F = -k(X - a) + mg \dots\dots (4)$




where the *positive* direction is downward:

$x = X - \left(a + \frac{mg}{k}\right)$

$\frac{mg}{k}$ is the change in displacement due to body weight.

$\therefore x = X - a - \frac{mg}{k}$

$X - a = x + \frac{mg}{k} \dots\dots (5)$

Sub. Eq. (5) in Eq. (4)

$\therefore F = -k\left(x + \frac{mg}{k}\right) + mg$

$F = -kx - \frac{kmg}{k} + mg$

This give again:

$\therefore F = -kx$

$-kx = m\ddot{x}$

$\therefore m\ddot{x} + kx = 0 \dots\dots (6)$

Eq. (6) is **second order** differential Eq. with constant coefficients of the harmonic oscillator or linear oscillator. As equation $y'' + py' + qy = 0$, so, to solve such equation we shall employ the ***trail method*** in which the function $(Ae^{qt})$ is the trail solution, where $(q)$ is a constant to be determined.

معادلة (6) هي معادلة تفاضلية من الدرجة الثانية ذات معاملات ثابتة لمذبذب توافقي أو مذبذب خطي. ويتم استخدام طريقة التجربة لحل المعادلة والذي فيه نجرب الحل $(Ae^{qt})$ حيث ان $(q)$ ثابت.

$x = Ae^{qt}$ …… (7)

$\frac{dx}{dt} = \dot{x} = Aqe^{qt}$ …… (8)

$\frac{d^2x}{dt^2} = \ddot{x} = A\, q^2\, e^{qt}$ …… (9)

Sub. Eq. (7) and (9) in Eq. (6)

$m\, q^2\, A\, e^{qt} + kA\, q^{qt} = 0 \quad ] \div A\, e^{qt}$

$mq^2 + k = 0$

$mq^2 = -k$

$q^2 = -\frac{k}{m}$

$\therefore q = \mp\sqrt{\frac{-k}{m}} = \mp i\sqrt{\frac{k}{m}}$

where $i = \sqrt{-1}$ , $\sqrt{\frac{k}{m}} = w_0$

$q = \mp i w_0$ …… (10)

Sub. Eq. (10) in Eq. (7)

$\therefore x = Ae^{\mp i\, w_0 t}$ …… (11)

For a linear differential eqns., solution are additive, so, the general solution is:

$x = A_+\, e^{iw_0 t} + A_-\, e^{-iw_0 t}$ …… (12)

using ***Euler's formula***

$e^{iu} = \cos u + i \sin u$

So we can rewrite Eq. (12) in the form:

$x = A_+(\cos w_0 t + i \sin w_0 t) + A_-(\cos w_0 t - i \sin w_0 t)$

$= A_+ \cos w_0 t + i\, A_+ \sin w_0 t + A_- \cos w_0 t - i\, A_- \sin w_0 t$

$= (A_+ + A_-) \cos w_0 t + (iA_+ - iA_-) sin w_0 t$

$x = a \sin w_0 t + b \cos w_0 t$ …… (13)

where $a = i\, A_+ - i\, A_-$ *and* $b = A_+ + A_-$

The real solution of Eq. (13) is:

$x = b \cos w_0 t$

or $x = A \cos(w_0 t + \theta_o)$ …… (14) ***Sinusoidal Oscillation of Displacement x*** where: $w_o$ is ***angular frequency*** 　　　　التردد الزاوي

$A$ is ***amplitude*** (the maximum value of $x$) 　　　　السعة (اعظم قيمة للازاحة)

Equation (14) represent cosine function for two angle

So, $A\cos(w_0 t + \theta_o) = A(\cos w_0 t \cos\theta_o + \sin w_0 t \sin\theta_o)$

Let $a = A\cos\theta_o$ , $b = A\sin\theta_o$

$$a^2 + b^2 = (A\cos\theta_o)^2 + (A\sin\theta_o)^2 = A^2(\cos^2\theta_o + \sin^2\theta_o)$$

$$A = (a^2 + b^2)^{1/2}$$

$$\frac{b}{a} = \frac{A\sin\theta_o}{A\cos\theta_o} = \tan\theta_o$$

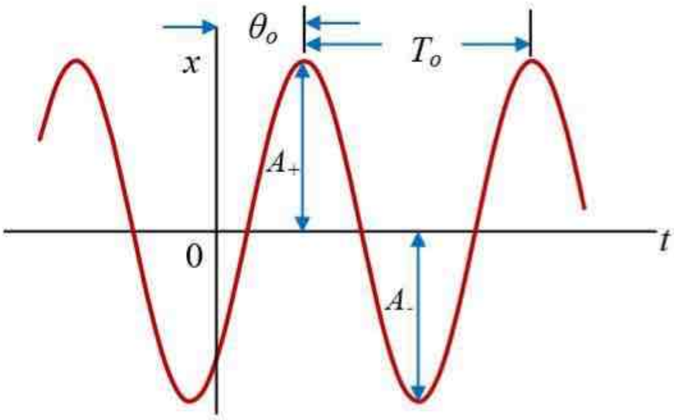
$$\theta_0 = \tan^{-1}\left(\frac{b}{a}\right) \ldots\ldots(15)$$ ***Initial Phase***

$T_o$ is **time period** of the oscillation (time required for one complete cycle); that is, the period is the time for which the product $(wt)$ increase by just $(2\pi)$

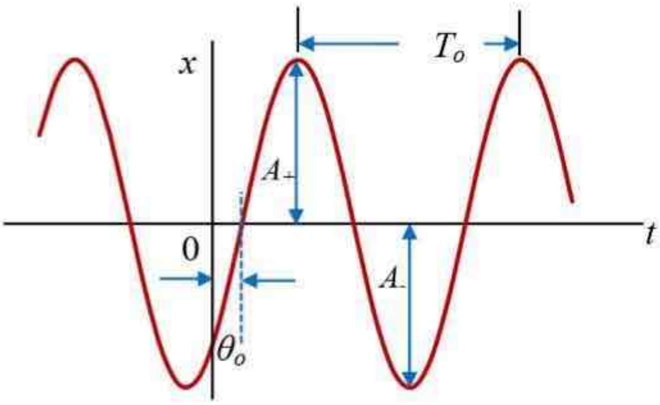
زمن الذبذبة الزمن اللازم لدورة كاملة

$$T_o = \frac{2\pi}{w_0} = 2\pi\sqrt{\frac{m}{k}} \ldots\ldots(16)$$

$$w_0 = 2\pi f_0 \ldots\ldots(17)$$




$x = A\cos(w_o t + \theta_o)$




$x = A\sin(w_o t + \theta_o)$

$f_0 \equiv$ Linear frequency of oscillation (is the number of cycles in unit time)

التردد الخطي للمتذبذب والذي يمثل عدد الدورات لوحدة الزمن

$$f_0 = \frac{1}{T_0} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \ldots\ldots(18)$$

**Example:**

A light spring is found to stretch an amount $b$ when it supports a block of mass $m$. If the block is pulled downward a distance $l$ from its equilibrium position and released at time $= 0$ , find the resulting motion as a function of $t$.

**Solution:**

In the static equilibrium

$$F = -kb = -mg$$

$$\therefore k = \frac{mg}{b}$$

$$w_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg}{mb}} = \sqrt{\frac{g}{b}}$$

Now find the constants for the equation of motion

$$x = A\cos(w_0 t + \theta_0)$$

$$at\ \ t = 0\ , \quad x = l \qquad \text{and } \dot{x} = 0$$

$$\dot{x} = -A\, w_0 \sin(w_0 t + \theta_0)$$

$$\dot{x} = -A\, w_0 \sin(w_0(0) + \theta_0) = 0$$

$$A\, w_0 \neq 0\ \ ,\ \ A = l\ \ is\ spring\ length, w_0\ is\ angular\ frequncy$$

$$A = l = X_{max}$$

$$\therefore \sin(\theta_0) = 0 \ \Rightarrow\ \theta_0 = 0$$

$$x = A\cos(w_0 t)$$

$$\therefore x = l\cos\left(\sqrt{\frac{g}{b}}t\right)$$

**3.3 Energy Consideration in Harmonic Motion**

Consider a particle moving under a linear restoring force$= -kx$. Let us calculate the work done by an external force $f_a$ in moving the particle from the equilibrium position $x = 0$ to some position $x$.

نفترض جسيم يتحرك تحت تأثيرقوة معيدة خطية $-kx$ نحسب الشغل المنجز بواسطة قوة خارجية $f_a$ لنقل الجسم من موضع التوازن $x = 0$ إلى موضع ما $x$

$$F = -kx \;\dots\dots (1)$$

$$F_a = -F = kx$$

$$\therefore W = \int F_a\, dx = \int_0^x kx\, dx = \frac{1}{2}kx^2$$

The work $W$ is stored in the spring as potential energy الشغل يخزن في النابض كطاقة كامنة

$$\therefore V(x) = W = \frac{1}{2}kx^2 = E_p \;\dots\dots (2)$$

***Total spring energy***

$$E = E_k + E_p$$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \;\dots\dots (3) \qquad * \frac{2}{m}$$

$$\frac{2E}{m} = \dot{x}^2 + \frac{k}{m}x^2$$

$$\dot{x}^2 = \frac{2E}{m} - \frac{k}{m}x^2$$

$$\dot{x} = \left(\frac{2E}{m} - \frac{k}{m}x^2\right)^{1/2} \;\dots\dots (4)$$

This can be integrated to give $(t)$ as function of $x$

$$\dot{x} = \frac{dx}{dt} \quad \Rightarrow dt = \frac{dx}{\dot{x}}$$

$$t = \int dt = \int \frac{dx}{\left(\frac{2E}{m} - \frac{k}{m}x^2\right)^{1/2}} \;\dots\dots (5)$$

➢ ***Derive the Equation of Time***

- ***When $x = A\cos\theta$***

Rewrite Eq.(5)

$$t = \int \frac{dx}{\left[\left(\frac{k}{m}\right)\left(\frac{2E}{k} - x^2\right)\right]^{1/2}} = \int \frac{dx}{\left(\frac{k}{m}\right)^{1/2}\left[\left(\frac{2E}{k} - x^2\right)\right]^{1/2}}$$

$$= \sqrt{\frac{m}{k}} \int \frac{dx}{\sqrt{\frac{2E}{k} - x^2}} = \sqrt{\frac{m}{k}} \int \frac{dx}{\sqrt{A^2 - x^2}} \;\dots\dots (6)$$

where $A = \sqrt{\frac{2E}{k}} \equiv$ amplitude

$$x = A\cos\theta \;\dots\dots (7)$$

$$\therefore x^2 = A^2 \cos^2\theta$$

$$\therefore A^2 - x^2 = A^2 - A^2\cos^2\theta$$

$$= A^2(1-\cos^2\theta)$$

$$A^2 - x^2 = A^2\sin^2\theta$$

$$\sqrt{A^2-x^2} = A\sin\theta \;\dots\dots (8)$$

From Eq.(7)

$$dx = -A\sin\theta\, d\theta \;\dots\dots (9)$$

Sub. Eqns.(8) and (9) in Eq. (6)

$$t = -\sqrt{\frac{m}{k}}\int \frac{A\sin\theta}{A\sin\theta}\, d\theta$$

$$t = -\sqrt{\frac{m}{k}}\int d\theta$$

$$t = -\sqrt{\frac{m}{k}}\theta + c \;\dots\dots (10)$$

$$\because\; x = A\cos\theta$$

$$\therefore \cos\theta = \frac{x}{A} \Rightarrow \theta = \cos^{-1}\left(\frac{x}{A}\right) \dots\dots (11)$$

Sub. Eq.(11) in Eq. (10)

$$t = -\sqrt{\frac{m}{k}}\cos^{-1}\left(\frac{x}{A}\right) + c \;\dots\dots (12)$$

- ***When*** $\boldsymbol{x = A\sin\theta}$

$$\therefore \sin\theta = \frac{x}{A} \Rightarrow \theta = \sin^{-1}\left(\frac{x}{A}\right) \dots\dots (a)$$

$$dx = A\cos\theta\, d\theta \;\dots\dots (b)$$

$$A^2 - x^2 = A^2 - A^2\sin^2\theta = A^2(1-\sin^2\theta)$$

$$A^2 - x^2 = A^2\cos^2\theta$$

$$\sqrt{A^2-x^2} = A\cos\theta \;\dots\dots (c)$$

Sub. Eqns. ($b$) and ($c$) in Eq. (6)

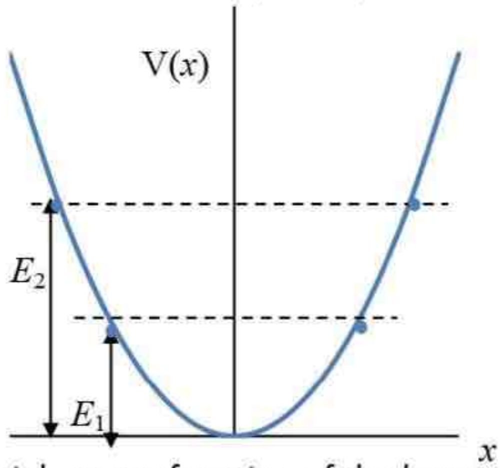
$$t = \sqrt{\frac{m}{k}} \int \frac{A\cos\theta}{A\cos\theta}\, d\theta$$

$$t = \sqrt{\frac{m}{k}}\,\theta + c \;\ldots\ldots (d)$$

Sub. Eq. $(a)$ in Eq. $(d)$

$$t = \sqrt{\frac{m}{k}} \sin^{-1}\left(\frac{x}{A}\right) + c \;\ldots\ldots (13)$$

The value of $x$ must lie between $\pm A \left(\pm\sqrt{\frac{2E}{k}}\right)$ in order for $\dot{x}$ to be real




Potential energy function of the harmonic oscillator.

**1)** at the upper point

$x = X_{max}\,, v = 0$

$$\therefore E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$E = 0 + \frac{1}{2}k$$

$$X_{max} = A$$

$$\therefore E = \frac{1}{2}kA^2 \rightarrow A = \sqrt{\frac{2E}{k}}$$

**2)** at the lower point

$x = 0\,, v = v_{max}$

$$\therefore E = \frac{1}{2}mv_{max}^2$$

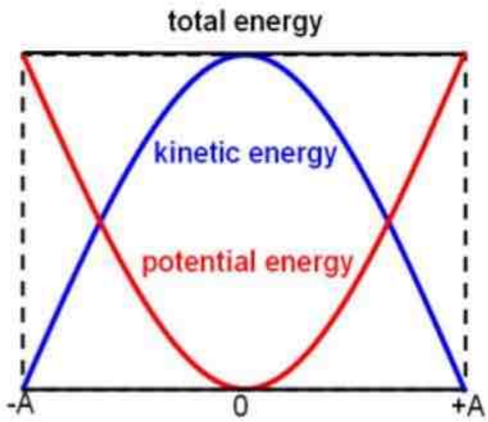
$$\frac{1}{2}m\,v_{max}^2 = \frac{1}{2}kA^2$$

$$v_{max}^2 = \frac{k\,A^2}{m}$$

$$v_{max} = \sqrt{\frac{k}{m}}A = Aw_0$$ ***Maximum Velocity for Harmonic Oscillator***

$$A = \frac{v_{max}}{w_0}$$

As the particle oscillates, the kinetic and potential energies continually change. The constant total energy is entirely in the form of ***kinetic energy at the center***, where $x = 0$ and $\dot{x} = \pm v_{max}$ and it is all ***potential energy at extrema***, where $x = \pm A$ and $\dot{x} = 0$.

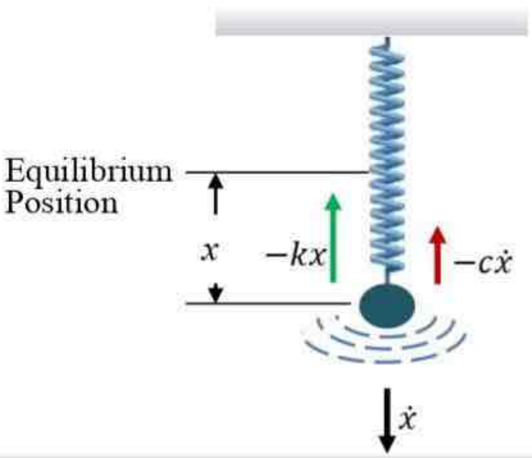



عندما يتأرجح الجسيم، تتغير الطاقة الكامنة والحركية باستمرار. تكون الطاقة الكلية ثابتة حيث تكون بشكل طاقة حركية في المركز و طاقة كامنة عند النهايات.

### 3.4 Damped Harmonic Motion

The foregoing analysis of the harmonic oscillator is *idealized* in that we didn't take into account *frictional forces*. These are always present in any mechanical system. Consider an object is supported by a spring of stiffness $k$ and there was a ***viscous retarding force*** varying linearly with the speed such as ***air resistance***.

التحليل السابق لمتذبذب توافقي مثالي لأنه لم ياخذ بنظر الاعتبار قوة الاحتكاك والتي تكون دائما موجودة في أي نظام ميكانيكي. لنفترض أن جسمًا ما معلقاً بنابض معامل مرونته $k$ وكانت هناك قوة معيقة لزجة متغيرة خطياً مع السرعة مثل مقاومة الهواء.



$F = -kx \dots\dots (1)$ $\quad$ $(\boldsymbol{restoring\ force})$

$F = -c\,\dot{x} \dots\dots (2)$ $\quad$ $(\boldsymbol{retarding\ force})$

Equation of motion then:

$$\therefore -kx - c\dot{x} = m\ddot{x} \quad \dots\dots(3)$$

$m\ddot{x} + c\dot{x} + kx = 0 \dots (4)$ ***Differential Eq. of Motion for Damped Harmonic Oscillator***

Use ***trail method*** to solve Eq. (4)

$$x = A\,e^{qt}$$

$$\therefore\ m\frac{d^2}{dt^2}(A\,e^{qt}) + c\frac{d}{dt}\,(A\,e^{qt}) + k\,A\,e^{qt} = 0$$

$$m\,q^2\,A\,e^{qt} + cq\,A\,e^{qt} + k\,A\,e^{qt} = 0] \div Ae^{qt}$$

$mq^2 + cq + k = 0 \quad \dots\dots(5)$ ***Auxiliary Equation***

$$ax^2 + bx + c = 0$$

$$x = \frac{-b\mp\sqrt{b^2 - 4ac_1}}{2a}$$

$$q = \frac{-c\mp\sqrt{c^2 - 4mk}}{2m} \dots\dots(6)$$

- $c^2 > 4mk$ **(*Over Damping*)**

  Here $q$ will be real and negative and the motion will be nonoscillatory $q_1 \neq q_2$ and $(x)$ decaying to zero exponentially with time.

المقدار $\boldsymbol{c^2 - 4mk}$ يحدد نوع التذبذب

حالة $\boldsymbol{c^2 > 4mk}$ تمثل حالة فوق التضاؤل وعندها $q$ تمتلك قيمتين حقيقيتين سالبتين مختلفتين لذلك تكون الحركة غير تذبذبية وتهبط فيها قيمة الازاحة $x$ أسياً الى الصفر مع الزمن

$$q = -\begin{cases}\gamma_1 \\ \gamma_2\end{cases} \quad \Rightarrow x = \begin{cases}A_1 e^{-\gamma_1 t} \\ A_2 e^{-\gamma_2 t}\end{cases}$$

The general solution for displacement is:

$$x = A_1\,e^{-\gamma_1 t} + A_2\,e^{-\gamma_2 t} \dots\dots(7)$$

- $c^2 = 4mk$ **(*Critical Damping*)**

  Here $q$ will be real also, and negative and the motion will be nonoscillatory. $(x)$ decaying to zero exponentially with time but in shorter time.

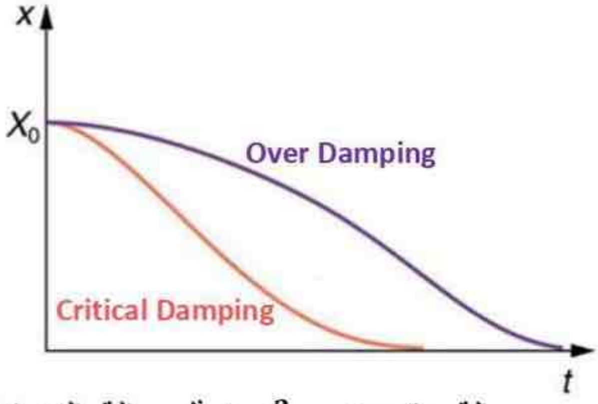
$$q_1 = q_2 = -\frac{c}{2m}$$

$$q = -\gamma \quad \Rightarrow x = \begin{cases}A_1 e^{-\gamma t} \\ A_2 t e^{-\gamma t}\end{cases}$$

The general solution for displacement is:

$$x = A_1\, e^{-\gamma t} + A_2 t\, e^{-\gamma t} \;\ldots\ldots (8)$$

$$x = e^{-\gamma t}(A_1 + tA_2) \;\ldots\ldots (9)$$




حالة $c^2 = 4mk$ تمثل حالة التضاؤل الحرج وعندها $q$ تمتلك قيمتين حقيقيتين سالبتين متساويتين لذلك تكون الحركة غير تذبذبية وتهبط فيها قيمة الازاحة $x$ أسياً الى الصفرمع الزمن

➢ $c^2 < 4mk$ **(*Under Damping*)**

Here $q$ will be complex; the imaginary part of its value gives an oscillatory motion.

$$q = \frac{-c \mp \sqrt{c^2 - 4mk}}{2m}$$

$$q = \frac{-c \mp \sqrt{\left(c^2 \cdot \frac{4m^2}{4m^2} - 4mk \cdot \frac{4m^2}{4m^2}\right)}}{2m}$$

$$q = \frac{-c \mp \sqrt{4m^2\left(\frac{c^2}{4m^2} - \frac{k}{m}\right)}}{2m} = \frac{-c \mp 2m\sqrt{\left(\frac{c^2}{4m^2} - \frac{k}{m}\right)}}{2m}$$

$$\boxed{\gamma = \frac{c}{2m} \to \gamma^2 = \frac{c^2}{4m^2} \quad , w_0 = \sqrt{\frac{k}{m}}}$$

$$q = \frac{-c \mp 2m\sqrt{\gamma^2 - w_0{}^2}}{2m}$$

$$q = -\frac{c}{2m} \mp \sqrt{\gamma^2 - w_0{}^2}$$

$$q_{1,2} = -\frac{c}{2m} + i\sqrt{w_0{}^2 - \gamma^2} = -\gamma \mp i\, w_1 \quad \textbf{\textit{Complex Conjugates Roots}}$$

where $w_1 = \sqrt{w_0{}^2 - \gamma^2}$ is ***Natural Frequency***

$$q_1 = -\gamma + i\, w_1$$

$$q_2 = -\gamma - i\, w_1$$

The Displacement then:

$$\therefore x = A_+\, e^{(-\gamma + i\, w_1)t} + A_-\, e^{(-\gamma - i\, w_1)t} \;\ldots\ldots (10)$$

$$\therefore x = e^{-\gamma t}\left(A_+\, e^{i\, w_1 t} + A_-\, e^{-i w_1 t}\right)$$

$$e^{iu} = \cos u + i \sin u \quad \textbf{\textit{Euler's Formula}}$$

$$x = e^{-\gamma t}\,[(i\,A_+ - i\,A_-)\sin w\,t + (A_+ + A_-)\cos w_1 t\,]$$

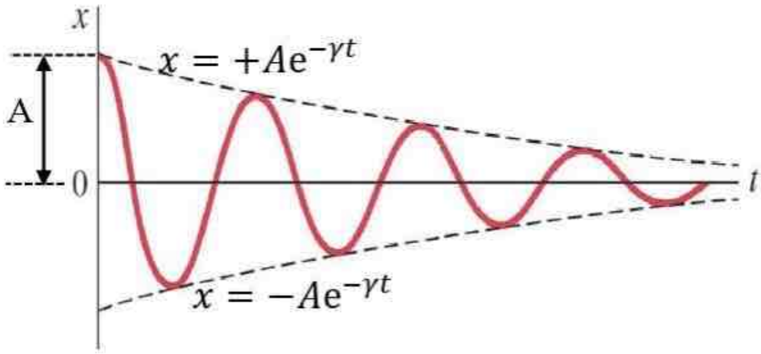
$$x = e^{-\gamma t}\,(\,a \sin w_1 t + b \cos w_1 t\,)$$

where $a = i\,(A_+ - A_-)$, $\quad b = A_+ + A_-$

or $x = A\,\mathrm{e}^{-\gamma t}\cos(w_1 t + \theta_0)$ ……. (11)

where $\theta_0 = \tan^{-1}\left(\frac{b}{a}\right)$

$$A = (a^2 + b^2)^{1/2}$$




Equation (11) shows that the ***two curves*** are given by $x = +A\mathrm{e}^{-\gamma t}$ and $x = +A\mathrm{e}^{-\gamma t}$ form an ***envelope*** of the curve of motion because the cosine factor takes on values between +1 and -1, including +1 and -1, at which points the curve of motion touches the envelope. Accordingly, the points of contact are separated by a time interval of ***one-half period***.

في حالة $c^2 < 4mk$ والتي تمثل حالة دون التضاؤل عندها نحصل على قيمتين غير حقيقيتين (خيالية) لـ$q$ والحركة هنا تكون تذبذبية والسعة تتضائل اسياً مع الزمن.

تظهر المعادلة (11) وجود منحنيتين هما $x = +A\mathrm{e}^{-\gamma t}$ و $x = +A\mathrm{e}^{-\gamma t}$ يشكلان غلافاً لمنحنى الحركة لأن عامل الجيب تمام يأخذ القيم بين +1 و -1 ، بضمنها +1 و -1 ، والتي يمس فيها منحنى الحركة ، الغلاف. وفقًا لذلك ، لذلك تنفصل نقاط التماس بفترة زمنية مقدارها نصف مدة الذبذبة.

### 3.5 Energy Consideration for Damped Harmonic Oscillator

The total energy of the damped harmonic oscillator is given by the sum of the kinetic and potential energies

الطاقة الكلية للمتذبذب التوافقي المضمحل في اية لحظة هي مجموع للطاقات الحركية والكامنة

$$E_t = E_k + E_p$$

$E_t = \frac{1}{2} m\, \dot{x}^2 + \frac{1}{2} k\, x^2 \ldots \ldots (1)$

To find ***time rate*** of change of $E_t$, we have to differentiate $E_t$ with respect to $t$:

$$\frac{dE_t}{dt} = \frac{1}{2} 2m\dot{x}\frac{d\dot{x}}{dt} + \frac{1}{2} 2kx\frac{dx}{dt}$$

$$= m\ddot{x}\dot{x} + k\dot{x}x$$

$$\frac{dE_t}{dt} = (m\ddot{x} + kx)\dot{x} \ldots \ldots (2)$$

We have the Eq. of motion for the damped harmonic oscillator

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$m\ddot{x} + kx = -c\dot{x} \ldots \ldots (3)$$

Sub. Eq. (3) in Eq. (2)

$$\therefore \frac{dE_t}{dt} = -c\dot{x}^2 \ldots \ldots (4)$$

This equation represents the rate at which the energy $E_t$ dissipated as frictional heat by virtue of the viscous resistance to the motion.

هذه المعادلة تمثل معدل تبدد الطاقة الكلية الى حرارة بسبب الاحتكاك وهي مقدار سالب دائماً

### 3.6 Forced Harmonic Motion (Resonance)

In this section, we study the motion of a damped harmonic oscillator that is subjected to a periodic driving force by an external agent.

Consider a damped harmonic oscillator motion affected by an external force ($F_{ext}$) that varying as a ***cosine*** wave with time $t$, the ***angular frequency*** $w$ and ***amplitude*** ($F_0$) such that

هنا ندرس حركة المتذبذب التوافقي المضمحل المدفوع بقوة خارجية توافقية. اي قوة تتغير بدالة جيبية مع الزمن

$$F_{ext} = F_0 \cos(wt + \theta) \ldots \ldots (1)$$

$$F_{ext} = F_0\, e^{i(wt+\theta)} \ldots \ldots (2)$$

There are three forces attached on the body:

1. Elastic restoring Force $= -kx$
2. The viscous damping force $= -c\dot{x}$
3. External force $= F_{ext}$

هناك ثلاث قوى مؤثرة في الجسم:

- قوة معيدة مرنة $(-kx)$
- قوة لزوجة مضمحلة $(-c\dot{x})$
- قوة خارجية $(F_{ext})$

عليه تكون القوة الكلية المؤثرة على الجسم مجموع لهذه القوى الثلاث

So, total force is:

$$\therefore -kx - c\dot{x} + F_{ext} = m\ddot{x} \ldots\ldots (3)$$

$$m\ddot{x} + c\dot{x} + kx = F_{ext} = F_0\, e^{i(wt+\theta)} \ldots\ldots (4)$$

Eq.(4) represent differential damped harmonic oscillator motion affected by an external force ($F_{ext}$ ). Suggested solution of this equation as:

معادلة (4) تمثل معادلة الحركة لمتذبذب توافقي مضمحل تحت تأثير قوة خارجية $(F_{ext})$

$$x = A\, e^{i(wt+\theta')} \ldots\ldots (5)$$

$$\dot{x} = \frac{dx}{dt} = \frac{d}{dt} A\, e^{i(wt+\theta')}$$

$$\dot{x} = i\, A\, w\, e^{i(wt+\theta')} = iwx \quad \ldots\ldots (6)$$

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{d^2}{dt^2} A\, e^{i(wt+\theta')}$$

$$\ddot{x} = i^2\, A\, w^2\, e^{i(wt+\theta')} = i^2 w^2 x = -w^2 x \ldots\ldots (7)$$

Sub. Eqns.(5) ,(6) and (7) in Eq. (4).

$$-mAw^2 e^{i(wt+\theta')} + cAwi\, e^{i(wt+\theta')} + kAe^{i(wt+\theta')} = F_0\, e^{i(wt+\theta)}\Big] * e^{-i(wt+\theta')}$$

$$\therefore\ -mA\, w^2 + iwcA + kA = F_0 e^{i(wt+\theta)} . e^{-i\,(wt+\,\theta')}$$

$$-mAw^2 + iwcA + kA = F_0\left[e^{iwt}\, e^{i\theta}\, e^{-iwt}\, e^{-i\theta'}\right]$$

$$-mAw^2 + iwcA + kA = F_0 e^{i(\theta-\theta')}$$

$$-mAw^2 + iwcA + kA = F_0[\cos(\theta-\theta') + i\sin(\theta-\theta')]$$

where $\varphi = (\theta - \theta') \equiv Phase\ difference\ (\boldsymbol{Phase\ angle})$

Separation between real and imaginary terms, we get:

$$-mA\, w^2 + kA = F_0 \cos(\theta-\theta') = F_0 \cos\varphi$$

$$iwcA = iF_0 \sin(\theta-\theta') = i\, F_0 \sin\varphi$$

$$A(k - mw^2) = F_0 \cos\varphi \ \ldots\ldots (8)$$

$$cwA = F_0 \sin\varphi \ldots\ldots (9)$$

Dividing Eq. (9) on Eq. (8)

$$\frac{cw}{k-mw^2} = \frac{F_0 \sin\varphi}{F_0 \cos\varphi} = \tan\varphi \ldots\ldots (10)$$

$$\frac{c}{2m} = \gamma \qquad \therefore \ \frac{c}{m} = 2\gamma$$

$$w_0 = \sqrt{\frac{k}{m}}$$

$$\therefore \tan\varphi = \frac{\frac{c}{m}w}{\frac{k}{m} - w^2}$$

$$\therefore \tan\varphi = \frac{2\gamma\, w}{{w_0}^2 - w^2} \ \ldots\ldots (11)$$

Squaring and adding Eqns. (8) and (9)

$$A^2(k - mw^2)^2 + c^2w^2A^2 = F_0^2(\cos^2\varphi + \sin^2\varphi) = F_0^2 \ \ldots\ldots (12)$$

$$A^2\,[(k - mw^2)^2 + c^2w^2] = F_0^2$$

$$A^2 = \frac{F_0^2}{(k - mw^2)^2 + c^2w^2}$$

$$\therefore A = \frac{F_0}{\sqrt{(k - mw^2)^2 + c^2w^2}} \ \ldots\ldots (13)$$

by dividing the numerator and denominator on $m$

$$A = \frac{\frac{F_0}{m}}{\sqrt{\left(\frac{k}{m} - \frac{mw^2}{m}\right)^2 + \frac{c^2w^2}{m}}}$$

So, in term of $\gamma$ and $w_0$

$$A = \frac{F_0/m}{\sqrt{\left(w_0^2 - w^2\right)^2 + 4\gamma^2 w^2}} \ \ldots\ldots (14)$$ ***Steady State Oscillation Amplitude***

Eq. (14) represent the amplitude ($A$) as a function of the driving frequency ($w$). The maximum value of amplitude valid only at ($w = w_0$) (***Resonance Frequency***). To find this frequency equal ***differential amplitude equation by zero.***

معادلة (14) تمثل السعة ($A$) كدالة للتردد الدافع ($w$). القيمة القصوى للسعة تتحقق فقط عند ($w = w_0$) (تردد الرنين).

$$\frac{dA}{dw} = \frac{d}{dw}\left[\frac{F_0/m}{\sqrt{\left(w_0^2 - w^2\right)^2 + 4\gamma^2w^2}}\right]$$

$$= \frac{F_0}{m}\,\frac{d}{dw}\,[(w_0^2 - w^2)^2 + 4\gamma^2w^2]^{-\frac{1}{2}}$$

$$= \frac{F_0}{m}\,\frac{d}{dw}\,[w_0^4 + w^4 - 2w_0^2\,w^2 + 4\gamma^2w^2]^{-\frac{1}{2}}$$

$$\frac{dA}{dw} = \frac{F_0}{m}\left(\frac{-1}{2}\right)\left[\, w_0^4 + w^4 - 2w_0^2\, w^2 + 4\gamma^2 w^2\right]^{-\frac{3}{2}} \cdot [0 + 4w^3 - 4w_0^2 w + 8\gamma^2 w]$$

$$= \frac{F_0}{m}\left(\frac{-1}{2}\right)\frac{4\, w^3 - 4w_0^2\, w + 8\gamma^2\, w}{\sqrt[3]{\left(w_0^2 - w^2\right)^2 + 4\gamma^2 w^2}}$$

$$= \frac{-F_0}{2m}\, \frac{4\, w^3 - 4w_0^2\, w + 8\gamma^2\, w}{\sqrt[3]{\left(w_0^2 - w^2\right)^2 + 4\gamma^2 w^2}}$$

$$\frac{dA}{dw} = \frac{-F_0}{2m}\, \frac{4\, w^3 - 4w_0^2\, w + 8\gamma^2\, w}{\sqrt[3]{\left(w_0^2 - w^2\right)^2 + 4\gamma^2 w^2}}] * \frac{-m}{2F_0\, w}$$

$$\frac{dA}{dw} = \frac{mF_0}{4mF_0\, w}\, \frac{4w\left(w^2 - w_0^2 + 2\gamma^2\right)}{\sqrt[3]{\left(w_0^2 - w^2\right)^2 + 4\gamma^2 w^2}} = \frac{\left(w^2 - w_0^2 + 2\gamma^2\right)}{\sqrt[3]{\left(w_0^2 - w^2\right)^2 + 4\gamma^2 w^2}}$$

$$\frac{dA}{dw} = 0$$

$$\therefore w^2 - w_0^2 + 2\gamma^2 = 0$$

$$w^2 = w_0^2 - 2\gamma^2$$

$$w = w_r = (w_0^2 - 2\gamma^2)^{1/2} \;\ldots\ldots (15)$$ ***Resonant Frequency Equation***

where $w_r \equiv$ ***resonant frequency*** for ***maximum amplitude***.

In case of weak damping, that is, when $c \ll 2\sqrt{mk}$ $or\ \gamma \ll w_0$

Then $w_0 \simeq w_r$

From Eq. (14) and (15) we can find $A_{max}$ in Resonant frequency.

$$w^2 = w_0^2 - 2\gamma^2 \;\ldots\ldots (16)$$

$$\therefore 2\gamma^2 = w_0^2 - w^2 \;\ldots\ldots (17)$$

Sub. Eq. (16) and (17) in Eq. (14)

$$A = \frac{F_0/m}{\sqrt{(2\gamma^2)^2 + 4\gamma^2(w_0^2 - 2\gamma^2)}}$$

$$A = \frac{F_0/m}{\sqrt{4\gamma^4 + 4\gamma^2 w_0^2 - 8\gamma^4}}$$

$$A = \frac{F_0/m}{\sqrt{4\gamma^2 w_0^2 - 4\gamma^4}}$$

$$A = \frac{F_0/m}{\sqrt{4\gamma^2(w_0^2-\gamma^2)}}$$

$$\therefore A_{max} = \frac{F_0/m}{2\gamma\sqrt{(w_0^2-\gamma^2)}} \; \ldots\ldots (18)$$

In other form:

$$A_{max} \simeq \frac{F_0}{2m\gamma\sqrt{(w_0^2-\gamma^2)}}$$

$$A_{max} \simeq \frac{F_0}{c\sqrt{(w_0^2-\gamma^2)}} \; \ldots\ldots (19)$$

$$\boxed{\frac{c}{2m} = \gamma \quad \therefore c = 2m\gamma}$$

In weak damping $\gamma \ll w_0$ then $\gamma^2$ can be ***neglect***

$$A_{max} \simeq \frac{F_0}{2\gamma m w_0} = \frac{F_0}{c w_0} \; \ldots\ldots (20)$$

$$F_0 \simeq 2A_{max}\gamma m w_0 = A_{max} c w_0$$

Sub. Eq. (20) in Eq. (14)

$$A = \frac{A_{max}\,\gamma}{\sqrt{(w_0-w)^2+\gamma^2}} \; \ldots\ldots (21)$$

when $|w_0 - w| = \gamma$

or $w = w_0 \mp \gamma$

Then

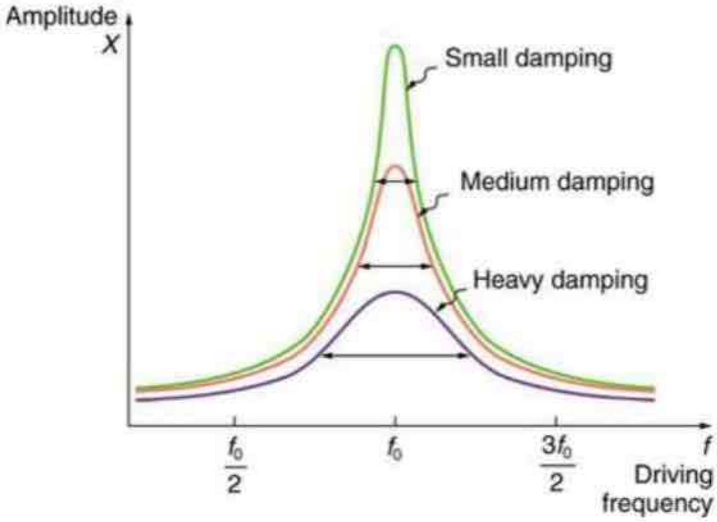
$$A = \frac{A_{max}\,\gamma}{\sqrt{(w_0-w_0+\gamma)^2+\gamma^2}}$$

$$A = \frac{A_{max}\,\gamma}{\sqrt{\gamma^2+\gamma^2}} = \frac{A_{max}\,\gamma}{\sqrt{2\gamma^2}} = \frac{A_{max}\,\gamma}{\sqrt{2}\,\gamma}$$

$$A^2 = \frac{1}{2}A^2{}_{max} \; \ldots\ldots (22)$$

This means that $\gamma$ is a measure of the width of the resonance curve. Thus, $2\gamma$ is the frequency difference between the points for which the energy is down by a factor of $\frac{1}{2}$ from the energy at resonance because the energy is proportional to $A^2$.

هذا يعني أن $\gamma$ هو مقياس لعرض منحنى الرنين. لذلك $2\gamma$ تمثل فرق التردد بين النقطتين اللتين تنخفض فيهما الطاقة بمقدار نصف طاقة الرنين لأن الطاقة تتناسب مع $A^2$




Another way of designating the sharpness of the resonance peak for the driven oscillator is in terms of the parameter ($Q$) called ***Quality Factor*** of the resonant system.

هناك طريقة أخرى لتعيين حدة قمة الرنين للمتبذب القسري وهي من خلال حساب المعامل ($Q$) الذي يسمى معامل النوعية للرنين.

$$Q = \frac{w_r}{2\gamma} \dots\dots (23)$$

In the case of weak damping

في حالة التضاؤل الضعيف

$$Q \approx \frac{w_0}{2\gamma} \dots\dots (24)$$

The total width $\Delta w$ at the half energy points is approximately

$$\Delta w = 2\gamma \approx \frac{w_0}{Q} \dots\dots (25)$$

$$w = 2\pi f$$

$$\therefore \frac{\Delta w}{w_0} = \frac{\Delta f}{f_0} \simeq \frac{1}{Q} \dots\dots (26)$$

giving the fractional width of the resonance peak,

العرض الجزئي لقمة الرنين

$Q = 10^4$ [ quartz oscillators ]

**Example:**

Determine the resonance frequency and the quality factor for the damped oscillator if the damping frequency $= \frac{w_0}{4}$ . Then find the phase angle $\theta$ if the applied frequency is $\frac{w_0}{2}$

**Solution:**

$$w_r = ({w_0}^2 - 2\gamma^2)^{1/2}$$

$$= ({w_0}^2 - \frac{2{w_0}^2}{16})^{1/2}$$

$$= w_0 \sqrt{\frac{7}{8}} = \sqrt{\frac{k}{m}} \sqrt{\frac{7}{8}}$$

$$Q = \frac{w_r}{2\gamma} = \frac{w_0\left(\frac{7}{8}\right)^{1/2}}{2\left(\frac{w_0}{4}\right)} = 2\sqrt{\frac{7}{8}} = 1.87$$

$$\tan\varphi = \frac{2\gamma\, w}{(w_0^2 - w^2)}$$

$$= \frac{2\,\frac{w_0}{4}\frac{w_0}{2}}{w_0^2 - (\frac{w_0}{2})^2} = \frac{2\,\frac{w_0^2}{8}}{w_0^2 - \frac{w_0^2}{4}}$$

$$= \frac{\frac{1}{4}w_0^2}{w_0^2\left(1 - \frac{1}{4}\right)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{4}\cdot\frac{4}{3} = \frac{1}{3}$$

$$\tan\varphi = \frac{1}{3} \Rightarrow \ \varphi = \tan^{-1}\left(\frac{1}{3}\right) = 18.5^0$$

## 4.1 Introduction

We now examine the general case of the motion of a particle in three dimensions. The vector form of the equation of motion for such a particle is

$\vec{F} = \frac{d\vec{P}}{dt}$ ……(1) (Newton's 2nd Law)

in which $\vec{P} = m\vec{v}$ is the linear momentum of the particle.

$\vec{F} = \frac{d(m\vec{v})}{dt}$ ……(2)

This vector equation is equivalent to ***three scalar equations*** in Cartesian coordinates.

هنا ندرس حركة الجسم بالابعاد الثلاثة. عليه كل متجه يوصف بدلالة ثلاث مركبات عددية في الاحداثيات الكارتيزية

$$\left.\begin{aligned} F_x &= \frac{d}{dt}(m\dot{x}) = m\ddot{x} \\ F_y &= \frac{d}{dt}(m\dot{y}) = m\ddot{y} \\ F_z &= \frac{d}{dt}(m\dot{z}) = m\ddot{z} \end{aligned}\right\} \quad …….(3)$$

لاتوجد طريقه عامة لايجاد الحلول لجميع الحالات الممكنه للقوة. لكن هناك انواع عديدة لدوال القوة يمكن حل معادلاتها التفاضليه بطرق بسيطه نسبياً.

## 4.2 The Work Principle

That means the ***work*** done on a particle causes it ***to gain or lose kinetic energy***.

هذا يعني أن الشغل المنجز على الجسيم يؤدي إلى اكتساب أو فقدان الطاقة الحركية.

Take the dot product of both sides of Eq.(1) with the velocity $\vec{v}$:

$$\vec{F} \cdot \vec{v} = \frac{d\vec{P}}{dt} \cdot \vec{v}$$

$$\vec{F} \cdot \vec{v} = \frac{d(m\vec{v})}{dt} \cdot \vec{v} \quad ……(3)$$

$$= m\frac{d\vec{v}}{dt} \cdot \vec{v} = m\frac{d}{dt}(\vec{v} \cdot \vec{v})$$

$$= m\left(\vec{v}\frac{d\vec{v}}{dt} + \vec{v}\frac{d\vec{v}}{dt}\right)$$

$$= 2m\vec{v}\frac{d\vec{v}}{dt}$$

$\because \frac{d}{dt}(\vec{v}\cdot\vec{v}) = 2\vec{v}\frac{d\vec{v}}{dt}$

$\therefore \; \frac{d\vec{v}}{dt} = \frac{\frac{d}{dt}(\vec{v}\cdot\vec{v})}{2\vec{v}}$

$\vec{v}\cdot\vec{v} = |v||v|\cos\theta = |\vec{v}|^2$

$\therefore \; \frac{dv}{dt} = \frac{d}{dt}\left(\frac{|\vec{v}|^2}{2\vec{v}}\right) = \frac{d}{dt}\left(\frac{\vec{v}}{2}\right)$ ......(4)

Multiply Eq.(4) by $(m)$

$\frac{d(m\vec{v})}{dt} = \frac{d}{dt}\frac{m\vec{v}}{2}$ ... ... (5)

Sub. Eq.(5) in Eq. (3)

$\vec{F}\cdot\vec{v} = \frac{d}{dt}\frac{m\vec{v}}{2}\cdot\vec{v}$

$\vec{F}\cdot\vec{v} = \frac{d}{dt}\left(\frac{1}{2}m\vec{v}^2\right)$

Where ( $T = \frac{1}{2}m\vec{v}^2$) is ***kinetic energy***

$\therefore \; \vec{F}\cdot\vec{v} = \frac{dT}{dt}$

$\vec{F}\cdot\vec{v}dt = dT$

$\vec{v} = \frac{dx}{dt} \;\Rightarrow\; \vec{v}dt = dx$

$\therefore \vec{F}\cdot d\vec{x} = dT$ (in one dimension)

In general (Three dimension) $\vec{r} = (x, y, z)$

$$\left.\begin{array}{ll}\vec{F}\cdot d\vec{r} = dT & (Differential\ Form)\\ \int\vec{F}\cdot d\vec{r} = \int dT & (Integral\ Form)\end{array}\right\} \text{......(6)} \quad \textbf{\textit{Work Equation}}$$

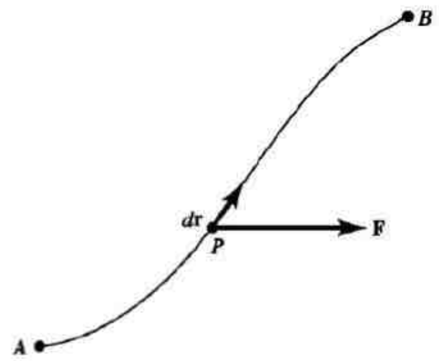
The equation states that the work done on the particle is equal to the increment in the kinetic energy.

- الشغل المنجز على جسيم يساوي صافي التغير في الطاقه الحركيه او يساوي الزيادة في الطاقه الحركية.
- معادلة (6) هي تكامل خطي يمثل الشغل المنجز على جسم من قبل القوة عندما يتحرك الجسم على طول مسار الحركة.

## 4.3 Conservative Forces and Force Fields

The line integral represents the work done on the particle by the force $\vec{F}$ as the particle moves along its trajectory from **A** to ***B***.

- التكامل الخطي يعتمد على مسار الحركة بمعنى ان الشغل المنجز يعتمد عادة على مسار الجسم من نقطه الى اخرى




The work done by a force $\vec{F}$ is the line integral $\int_A^B \vec{F} \cdot d\vec{r}$

Eq. (6) states that the work done on a particle by the net force acting on it, in moving from one position in space to another, is equal to the ***difference in the kinetic energy*** of the particle at these two positions, which mean that the work done on the particle depend on the ***particle path*** in the space. This required ***detailed knowledge*** of the motion of the particle from ***A*** to ***B*** to calculate the work done on it by the force.

In the case of a special type of force called conservative force, the ***work*** done by a force in moving a particle from point ***A*** to point ***B*** equal to the ***difference in the kinetic energy*** of the particle at these two positions. Many of the physically important forces are conservative type.

- أذا كانت $F$ دالة للموقع فقط عندها يكون لدينا مجال قوة استاتيكية (static force field) وهذه القوة تكون محفوظة.
- رياضياً المجال المحفوظ هو المجال الذي يعبر عنه بالعلاقه $\vec{F}.d\vec{r}$
- فاذا كان الجسيم يتحرك في مجال محفوظ فيمكن حساب الشغل من تكامل المقدار $\int \vec{F}.\vec{d}\ \vec{r}$ وبالتالي يمكن ان نحسب مقدار الزيادة في الطاقه الحركية.
- كون التكامل الخطي هنا يعتمد على المسار بين النقطتين $A$ و $B$ ، وهذا يتطلب معرفة تفصيلية لحركة الجسيم من النقطة $A$ إلى النقطة $B$ لحساب الشغل المنجز عليه من قبل القوة.
- عليه ستكون المعادلة فعالة في حالة القوى المحافظة فقط، ولكون اغلب القوى المؤثرة حولنا هي من النوع المحافظة فلن نجد مشكلة في حساب الشغل المنجز من التكامل $\int \vec{F} \cdot \vec{d}\ \vec{r}$

The tables below represent a comparison between conservative and non conservative forces with examples for each one of them.

| Conservative forces | Non conservative forces |
|---|---|
| The force is called conservative if work done by force is dependent only initial and final position of body ***not depends on path followed by body***. | The force is called non conservative force if work is done by the force is ***depend on path followed by body***. |
| The work done by conservative force in ***close path is zero***. | The work done by non-conservative force in a ***close path is not zero***. |

| Conservative forces | Non conservative forces |
|---|---|
| Gravitational forces | Frictional forces |
| Magnetic force | Viscous forces |
| Elastic spring force | Air resistance force |
| Electric force | Tension in a string |
| | Propulsion force of the rocket |

## 4.4 Potential Energy Function

The work integral in Cartesian coordinates given by:

$$\int F \cdot d\vec{r} = \int \left[ F_x \, dx + F_y \, dy + F_z \, dz \right] \dots \dots (1)$$

$$\left.\begin{aligned} F_x &= -\frac{\partial V}{\partial x} \\ F_y &= -\frac{\partial V}{\partial y} \\ F_z &= -\frac{\partial V}{\partial z} \end{aligned}\right\} \dots\dots\dots\dots\dots(2)$$

where $V(x, y, z)$ is ***potential energy function*** (scalar function)

So, Eq.(1) become:

$$\int \vec{F} \cdot d\vec{r} = \int [ -\frac{\partial V}{\partial x} dx - \frac{\partial V}{\partial y} dy - \frac{\partial V}{\partial z} dz]$$

$$\int \vec{F} \cdot d\vec{r} = -\int dV$$

$$\therefore \int \vec{F} \cdot d\vec{r} = -\int dV \quad \dots.(3)$$

By comparing with Eq.(6) in pervious section

$$\therefore \int d\,T = - \int dV$$

وهذا يدل على ان كل $T$ و $-V$ يختلف عن الاخر بكمية ثابتة

So, a general conservation of total energy principle:

$$T + V = Constant = E$$

$$\frac{1}{2} m \, v^2 + V_{(x,y,z)} = E \dots \dots (4)$$

When a particle moves in a ***conservative field*** of a force the ***sum of kinetic and potential*** energies remains ***constant*** throughout the motion.

عندما يتحرك الجسم في مجال محافظ للقوة فان مجموع الطاقه الحركيه والكامنه يبقى ثابت خلال الحركة.

## 4.5 The Potential in a Uniform Gravitational Field

In the case of a projectile moving in a uniform field of force such as a particle acted upon by gravity near the surface of the earth, only the ***gravity*** acts on the projectile, vertical motion. Choosing the z-axis to be vertical, the following equation of motion:

اذا تحرك جسيم بتأثير مجال قوة منتظم مثل حركة جسيم بتأثير قوة الجاذبيه الارضية قرب سطح الارض. فأذا كان الاتجاه العمودي بأتجاه محور $Z$ فان مقدار القوة $(mg)$ (قوة الجاذبية) يكون بأتجاه محور $z$ السالب، فاذن يجب ان تحقق دالة الجهد المعادلة (1).

$$\left.\begin{aligned} F_x &= -\frac{\partial V}{\partial x} = 0 \\ F_y &= -\frac{\partial V}{\partial y} = 0 \\ F_z &= -\frac{\partial V}{\partial z} = -mg \end{aligned}\right\} \quad \ldots\ldots(1)$$

$$\therefore \int dV = mg \int dz$$

$V_{(x,y,z)} = mgz + c$ (***Potential Function***)

where $c$ is an arbitrary constant and it is equal to zero at the earth's surface. So the energy equation becomes:

ثابت التكامل $c$ هو ثابت اعتباطي وقيمته تساوي صفر عند سطح الارض

$$\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz = E \ldots\ldots(2)$$

So, for any given case the total energy can be calculated from the knowledge of the initial conditions of the motion.

لأي حالة معينة، يمكن حساب الطاقة الكلية من معرفة الشروط الأبتدائية للحركة.

## 4.6 Conditions for the Existence of a Potential Function

One dimensional motion of a particle is always ***conservative*** if the force as a function of ***position only***. That is, if we have a force, $F(x)$, which is only a function of position, then $F(x)\,dx$ is always a perfect differential. This means that we can define a potential function as

$$V = -\int F \cdot dx$$

الحركة في خط مستقيم (بعد واحد) تكون دائماً محافظة اذا كانت القوه دالة للموقع فقط. عليه يمكن حساب دالة الجهد من التكامل $V = -\int F \cdot dx$

In two and three dimensions, we would, in principle, expect that any force which depends only on position, $F(r)$, to be conservative. In general, this is ***not sufficient***, unless satisfy certain criteria does a potential function exits

اما في حالة الحركة ببعدين او ثلاثه ابعاد نتوقع ان تكون اي قوة دالة للموقع فقط تكون محافظة. بشكل عام، هذا لا يكفي، إلا إذا استوفى شروط معينة عليه تكون دالة الجهد موجودة.

$$\left.\begin{array}{l} F_x = F_x(x, y, z) \\ F_y = F_y(x, y, z) \\ F_z = F_z(x, y, z) \end{array}\right\} \quad \dots\dots\dots (1)$$

Assume that a potential function is exist:

نفترض دالة الجهد موجودة

$$\left.\begin{array}{l} F_x = -\dfrac{\partial V}{\partial x} \\ F_y = -\dfrac{\partial V}{\partial y} \\ F_z = -\dfrac{\partial V}{\partial z} \end{array}\right\} \quad \dots\dots\dots\dots (2)$$

If we take the partial derivative of $F_x$ with respect to $y$

نأخذ المشتقة الجزئية لـ $F_x$ بالنسبة لـ $y$

$$\frac{\partial F_x}{\partial y} = -\frac{\partial^2 V}{\partial y\, \partial x} \dots\dots (3)$$

$$\frac{\partial F_y}{\partial x} = -\frac{\partial^2 V}{\partial x\, \partial y} \dots\dots (4)$$

$$\because \ \frac{\partial^2 V}{\partial y\, \partial x} = \frac{\partial^2 V}{\partial x\, \partial y} \dots\dots (5)$$

$$\therefore \frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \dots\dots\dots . (6)$$

الطرف الايمن من المعادلتين (3) و (4) متساوي لان $V$ دالة مستمرة وكذلك مشتقتها الاولى والثانيه

A similar argument can be made with the pairs $(F_y, F_z), (F_x, F_z)$

$$\left.\begin{array}{l} \dfrac{\partial F_x}{\partial y} = \dfrac{\partial F_y}{\partial x} \\ \dfrac{\partial F_y}{\partial z} = \dfrac{\partial F_z}{\partial y} \\ \dfrac{\partial F_x}{\partial z} = \dfrac{\partial F_z}{\partial x} \end{array}\right\} \quad \dots\dots\dots . (7)$$

These are the necessary conditions, on $F_x, F_y$ and $F_z$ for a potential function to exist. They express the condition that

$\vec{F} \cdot d\vec{r} = F_x\, dx + F_y\, d_y + F_z\, d_z$ is an exact differential.

Then the force components are indeed derivable from a potential function, and the sum of the kinetic energy and potential energy is constant.

فاذا كانت المعادلات في (7) صحيحة فان مركبات القوه تكون فعلا مشتقه من دالة الجهد $V_{(x,y,z)}$ ويكون مجموع الطاقه الحركيه والكامنه مقداراً ثابتاً. وهذه المعادلات تدعى بشروط تحقق (exist) دالة الجهد.

### 5.7 Potential for the Inverse (Square Law of Force)

Gravitational force varies inversely as square of the distance measured from the earth's center. This inverse - square relation is also found to be the law of force for electric fields of elementary particles.

**(جهد قانون التربيع العكسي )**

قوة الجذب بين جسمين تتناسب عكسياً مع مربع المسافه بينهما وفي حالة الجاذبيه الارضيه (في مجال الارض) فانها تتناسب مع مربع المسافه المقاسة من مركز الارض.

ان علاقة التربيع العكسي هذه هي ايضاً تمثل قانون قوة مجال التنافر والتجاذب بين الشحنات الكهربائية.

The analytical form for the inverse – square law can be written as:

$\vec{F}\ \alpha - \frac{1}{r^2}$

$\vec{F} = -k\frac{\vec{n}}{r^2}$ ...... (1) ( The inverse – square law )

Where $\vec{n}$: unit vector in the direction of the radius vector $\boldsymbol{r}$

$k$: constant proportionality

The negative sign indicates that the force is attractive or pointing toward the origin.

While positive sign denote a repulsive force pointing away from the origin.

$\vec{n}$ : الوحدة الاتجاهيه للازاحة (نصف القطر اي باتجاه قيمه نصف القطر $\vec{r}$)

$k$ : ثابت التناسب

والاشارة السالبه دلالة على ان القوه هي تجاذبيه متجه نحو نقطه الاصل.

وتكون الاشارة موجبة عندما تكون القوه تنافرية مبتعدة عن نقطه الاصل.

$\because\ \vec{n} = \frac{r}{|\vec{r}|} = \frac{\vec{r}}{r} \dots\dots (2)$

Thus the inverse-square law can also be written as:

$\therefore \vec{F} = -k\frac{\vec{r}}{r^3} \quad \dots\dots (3)$ (Inverse square law of force )

In Cartesian coordinate

$\therefore\ \vec{r} = ix + jy + kz \dots\dots (4)$

and

$r = (x^2 + y^2 + z^2)^{1/2} \dots\dots (5)$

So when Sub. Eqns. (4) and (5) in eqn. (3)

$\vec{F} = -k\,(ix + jy + kz)(x^2 + y^2 + z^2)^{-3/2} \dots\dots (6)$ Inverse-square law in rectangular coordinates

Consider that the potential function:

$V(r) = -\frac{k}{r} \dots\dots\dots (7)$

Gives the correct force, that is:

$F(r) = -\frac{dV}{dr} = -\frac{d}{dr}\left(-\frac{k}{r}\right) = -\frac{k}{r^2}$ [ one dimension ]

In three dimensions Eqn. (7) rewrite as:

$$V_{(x,y,z)} = \frac{-k}{(x^2+y^2+z^2)^{1/2}}$$

$$V_{(x,y,z)} = -k\,(x^2 + y^2 + z^2)^{-1/2}$$

Then the force components needed to give the force function

$$\left.\begin{aligned} F_x &= -\frac{\partial V}{\partial x} = -kx(x^2 + y^2 + z^2)^{-3/2} \\ F_y &= -\frac{\partial V}{\partial y} = -ky(x^2 + y^2 + z^2)^{-3/2} \\ F_z &= -\frac{\partial V}{\partial z} = -kz(x^2 + y^2 + z^2)^{-3/2} \end{aligned}\right\} \dots\dots\dots (8)$$

**Note:** $k$ here is constant gravity while in the case of vibrational motion they were constant elastic (***Stiffness Constant***)

**ملاحظة:** $k$ هنا يمثل ثابت الجاذبيه بينما في حالة الحركة الاهتزازيه فانه يمثل ثابت المرونه

### 4.8 The Del Operator

If the force field is conservative so that the components are given by the partial derivative of potential energy function.

اذا كان مجال القوه محافظاً عندها يمكن لمركبات القوة أن تعطى بدلالة المشتقات الجزئية لدالة الطاقة الكامنة.

We can now express a conservative force $F$ vectorially as:

$$\vec{F} = -i\,\frac{\partial V}{\partial x} - j\,\frac{\partial V}{\partial y} - k\,\frac{\partial V}{\partial z}$$

$$\vec{F} = -\nabla V \;\ldots\ldots (1)$$

Where: $\vec{\nabla}$ is del operator given as:

$$\vec{\nabla} = i\,\frac{\partial}{\partial x} + j\,\frac{\partial}{\partial y} + k\,\frac{\partial}{\partial z} \;\ldots\ldots (2)$$

1. $\vec{\nabla}V = \text{Gradient } V \text{ or } (\text{grad } V)$
   - Mathematically, the gradient of a function is a vector that represents the maximum spatial derivative of the function in direction and magnitude.
   - Physically, the negative gradient of the potential energy function gives the direction and magnitude of the force that acts on a particle located in a field created by other particles.
   - The meaning of the negative sign is that the particle is urged to move in the direction of decreasing potential energy rather than in the opposite direction.

- $\nabla V$ يسمى بانحدار $V$.
- رياضياً يعني التفاضل الموضعي للدالة في المقدار والاتجاه .
- فيزيائياً يعني أن الانحدار السالب لدالة الطاقه الكامنه يعطي اتجاه ومقدار القوة التي تؤثر على جسيم موضوع في مجال ناتج عن جسيمات اخرى.
- الاشارة السالبه تعني ان الجسيم اجبر على الحركة باتجاه تناقص الطاقه الكامنه بدلاً من الاتجاه المعاكس.

**2.** $\vec{\nabla} \times F = \text{Curl } \vec{f}$

- $\vec{\nabla} \times \vec{F}$ يسمى بدوران (التفاف) متجه القوة $\vec{F}$

The condition that a force be conservative can be written compactly as

$\vec{\nabla} \times \vec{F} = 0 \ldots\ldots (4)$ $(\text{Then The Force } \vec{\text{F}} \text{ is Conservative})$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

**3.** $\vec{\nabla} \cdot \vec{F} = \text{divergence of } \vec{F}$

$(\vec{\nabla} \cdot \vec{F})$ is called the divergence of $\vec{F}$ which gives a measure of the density of the sources of the field at a given point, which is of particular importance in the theory of electricity and magnetism.

$(\vec{\nabla} \cdot \vec{F})$ يمثل تفرق (تباعد) $\vec{F}$ وهي مقياس لكثافة المجال في نقطه معينه وهي مهمة في النظريه الكهربائية والمغناطيسية.

**Example 1:** Find the gradient of scaler $f = x^3 - 2xy^5 + y^2z^3$

**Solution:**

$$\vec{\nabla} f = \boldsymbol{i}\frac{\partial f}{\partial x} + \boldsymbol{j}\frac{\partial f}{\partial y} + \boldsymbol{k}\frac{\partial f}{\partial z}$$

$$\vec{\nabla} f = \boldsymbol{i}\frac{\partial( x^3-2xy^5+y^2z^3)}{\partial x} + \boldsymbol{j}\frac{\partial( x^3-2xy^5+y^2z^3)}{\partial y} + \boldsymbol{k}\frac{\partial( x^3-2xy^5+y^2z^3)}{\partial z}$$

$$\vec{\nabla} f = \boldsymbol{i}\frac{\partial}{\partial x}( x^3 - 2xy^5) + \boldsymbol{j}\ \frac{\partial}{\partial y}(-2xy^5 + y^2z^3) + \boldsymbol{k}\frac{\partial}{\partial z}(y^2z^3)$$

$$\vec{\nabla} f = \boldsymbol{i}(3x^2 - 2y^5) + \boldsymbol{j}(-10xy^4 + 2yz^3)\ + \boldsymbol{k}(3y^2z^2)$$

**Example 2:** Find the curl of vector $\vec{A} = x^2y\boldsymbol{i} + xyz\boldsymbol{j} - x^2y^2\boldsymbol{k}$

**Solution:**

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xyz & -x^2y^2 \end{vmatrix}$$

$$= \boldsymbol{i}\left(\frac{\partial}{\partial y}(-\,x^2y^2) - \frac{\partial}{\partial z}(xyz)\right) - \boldsymbol{j}\left(\frac{\partial}{\partial x}(-x^2y^2) - \frac{\partial}{\partial z}(x^2y)\right) + \boldsymbol{k}\left(\frac{\partial}{\partial x}(xyz - \frac{\partial}{\partial y}(x^2y)\right)$$

$$= \boldsymbol{i}\big(-2\,x^2y - xy\big) - \boldsymbol{j}(-2xy^2 - 0) + \boldsymbol{k}\big(yz - x^2\big)$$

$$\vec{\nabla} \times \vec{A} = \boldsymbol{i}(-2\,x^2y - xy) + \boldsymbol{j}(2xy^2) + \boldsymbol{k}(yz - x^2)$$

**Example 3:** Find the divergence of vector $\vec{A} = x^2y\boldsymbol{i} + xyz\boldsymbol{j} - x^2y^2\boldsymbol{k}$

**Solution:**

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-x^2y^2)$$

$$\vec{\nabla} \cdot \vec{A} = 2xy + xz + 0$$

$$\vec{\nabla} \cdot \vec{A} = 2xy + xz$$

**Example 4:** Is the force field $\vec{F} = \boldsymbol{i}xy + \boldsymbol{j}xz + \boldsymbol{k}yz$ conservative?

**Solution:**

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xz & yz \end{vmatrix}$$

$$\vec{\nabla} \times \vec{F} = \boldsymbol{i}\left(\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(xz)\right) - \boldsymbol{j}\left(\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial z}(xy)\right) + \boldsymbol{k}\left(\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(xy)\right)$$

$$\vec{\nabla} \times \vec{F} = \boldsymbol{i}(z - x) - \boldsymbol{j}(0 - 0) + \boldsymbol{k}(z - x)$$

$$\vec{\nabla} \times \vec{F} = \boldsymbol{i}(z - x) + \boldsymbol{k}(z - x)$$

$\therefore \vec{\nabla} \times \vec{F} \neq 0 \quad \rightarrow \vec{F}$ is non conservative. النتيجه لا تساوي صفر، بالتالي المجال غير محافظ

**Example 5:** For what values of the constants $a$, $b$ and $c$ is the force

$$\vec{F} = \boldsymbol{i}(ax + by^2) + \boldsymbol{j}cxy \text{ conservative?}$$

**Solution:**

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax + by^2 & cxy & 0 \end{vmatrix}$$

$$= \boldsymbol{i}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(cxy)\right) - \boldsymbol{j}\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(ax + by^2)\right) + \boldsymbol{k}\left(\frac{\partial}{\partial x}(cxy) - \frac{\partial}{\partial y}(ax + by^2)\right)$$

$$= \boldsymbol{i}(0 - 0) - \boldsymbol{j}(0 - 0) + \boldsymbol{k}(cy - 2by)$$

$$\vec{\nabla} \times \vec{F} = k(c - 2b)y$$

For conservative force must $\vec{\nabla} \times \vec{F} = 0$

$\therefore c - 2b = 0$

$c = 2b$

So $\vec{F}$ be conservative when $c = 2b$

**Example 6:** Find the force field of the potential Function $V = x^2 + xy + xz$.

**Solution:**

$$\vec{F} = -\vec{\nabla} V$$

$$\vec{F} = -\left(\boldsymbol{i}\frac{\partial V}{\partial x} + \boldsymbol{j}\frac{\partial V}{\partial y} + \boldsymbol{k}\frac{\partial V}{\partial z}\right)$$

$$\vec{F} = -\boldsymbol{i}\frac{\partial V}{\partial x} - \boldsymbol{j}\frac{\partial V}{\partial y} - \boldsymbol{k}\frac{\partial V}{\partial z}$$

$$\vec{F} = -\boldsymbol{i}\frac{\partial(x^2+xy+xz)}{\partial x} - \boldsymbol{j}\frac{\partial(x^2+xy+xz)}{\partial y} - \boldsymbol{k}\frac{\partial(x^2+xy+xz)}{\partial z}$$

$$\vec{F} = -\boldsymbol{i}\frac{\partial}{\partial x}(x^2 + xy + xz) - \boldsymbol{j}\frac{\partial}{\partial y}xy - \boldsymbol{k}\frac{\partial}{\partial z}xz$$

$$\vec{F} = -(2x + y + z)\boldsymbol{i} - \boldsymbol{j}x - \boldsymbol{k}x$$

Test If $\vec{F}$ is conservative or not

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -(2x+y+z) & -x & -x \end{vmatrix}$$

$$\vec{\nabla} \times \vec{F} = \boldsymbol{i}\left(\frac{\partial}{\partial y}(-x) - \frac{\partial}{\partial z}(-x)\right) - \boldsymbol{j}\left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial z}(-(2x+y+z))\right) + \boldsymbol{k}\left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(-(2x+y+z))\right)$$

$$\vec{\nabla} \times \vec{F} = \boldsymbol{i}(0-0) - \boldsymbol{j}(-1+1) + \boldsymbol{k}(-1+1) = 0$$

**Example 7:** Find the potential energy for harmonic oscillator: a) two dimensions b) three dimensions

**Solution:**

**a.** Two dimensions

$$\vec{F} = -k_1 x\boldsymbol{i} - k_2 y\boldsymbol{j}$$

Conservative force $\therefore \vec{F} = -\nabla V$

Test If $\vec{F}$ is conservative or not

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -k_1 x & -k_2 y & 0 \end{vmatrix}$$

$$\vec{\nabla} \times \vec{F} = \boldsymbol{i}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-k_2 y)\right) - \boldsymbol{j}\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(-k_1 x)\right) + \boldsymbol{k}\left(\frac{\partial}{\partial x}(-k_2 y) - \frac{\partial}{\partial y}(-k_1 x)\right)$$

$$\vec{\nabla} \times \vec{F} = \boldsymbol{i}(0-0) - \boldsymbol{j}(0-0) + \boldsymbol{k}(0-0) = 0$$

$$\vec{F} = -\vec{\nabla} V = -\frac{dV_x}{dx}\boldsymbol{i} - \frac{dV_y}{dy}\boldsymbol{j}$$

$\therefore \frac{dV_x}{dx} = k_1 x \Rightarrow dV_x = k_1 x dx$ *for x-coordinate*

$$V_x = \int k_1 x\, dx = \frac{1}{2}k_1 x^2$$

$\frac{dV_y}{dy} = k_2 y \Rightarrow dV_y = k_2 y dy$ *for y- coordinate*

$$V_y = \int k_2 y dy = \frac{1}{2}k_2 y^2$$

$$V = V_x + V_y$$

$$\therefore V = \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2$$

**b.** Three dimensions

$$F = -k_1 x\boldsymbol{i} - k_2 y\boldsymbol{j} - k_3 z\boldsymbol{k}$$

Conservative force $\therefore \vec{F} = -\vec{\nabla} V$

$$\vec{F} = -\vec{\nabla} V = -\frac{dV_x}{dx}\boldsymbol{i} - \frac{dV_y}{dy}\boldsymbol{j} - \frac{dV_z}{dz}\boldsymbol{k}$$

$\therefore \frac{dV_x}{dx} = k_1 x \Rightarrow dV_x = k_1 x dx$ *for x-coordinate*

$$V_x = \int k_1 x dx = \frac{1}{2}k_1 x^2$$

$\frac{dV_y}{dy} = k_2 y \Rightarrow dV_y = k_2 y dy$ *for y-coordinate*

$V_y = \int k_2 y dy = \frac{1}{2} k_2 y^2$

$\frac{dV_z}{dz} = k_3 z \Rightarrow dV_z = k_3 z dz$ *for z-coordinate*

$V_z = \int k_3 z dz = \frac{1}{2} k_3 z^2$

$V = V_x + V_y + V_z$

$\therefore V = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 y^2 + \frac{1}{2} k_3 z^2$

## 5.1 Introduction

It is very convenient in describing the motion of a particle, to use a coordinate system which, itself, is moving. A coordinate system fixed the motion of a projectile, although the earth is moving and rotating.

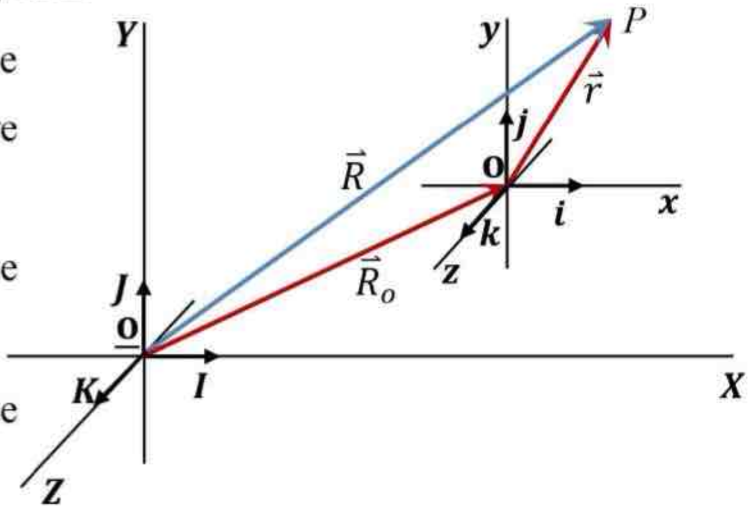
من الملائم جداً وصف حركة الجسيم، باستخدام نظام إحداثيات متحرك بحد ذاته. على سبيل المثال نظام الإحداثيات المستخدم لوصف حركة قذيفة مرتبط بالارض والتي تكون في حالة حركة انتقالية ودورانية في نفس الوقت (الأرض تتحرك وتدور بالوقت ذاته).

## 5.2 Translation of the Coordinate System

The simplest type of motion of the coordinate system is that of pure translation. In figure

$\underline{\boldsymbol{O}}\boldsymbol{XYZ}$: the primary coordinate system (assumed fixed).

$\boldsymbol{Oxyz}$: the moving coordinate system.




In the case of pure translation, the respective axes $\underline{O}X$ and $Ox$, and soon, remain parallel

$\underline{O}X \parallel Ox$
$\underline{O}Y \parallel Oy$
$\underline{O}Z \parallel Oz$

$P$: is the particle position.

$\vec{R}$: is the position vector of the particle $P$ in the fixed system.

$\vec{r}$: is the position vector of the particle in the moving system.

$\vec{R}_0$: is the displacement of the moving origin $\underline{O}O$

حركة المحاور الانتقاليه هي ابسط انواع الحركات.

يمكن وصف حركة جسيمه مثل $P$ بالنسبة لمنظومة المحاور الثابته ( $\underline{O}XYZ$) ومنظومة المحاور المتحركة ($Oxyz$).

في حالة كون حركة المحاور انتقالية فقط عندها كل محور في منظومة المحاور الثابتة يوازي المحور المقابل له في منظومة المحاور المتحركة.

$\vec{R}$ : متجه موضع الجسيم $P$ في المحاور الثابتة
$\vec{r}$ : متجه موضع الجسيم $P$ في المحاور المتحركة
$\vec{R}_0$ : ازاحة نقطة الاصل المتحركة oo

$$\vec{R} = \vec{r} + \vec{R}_o \quad \dots\dots\dots (1)$$

Taking the first and second time derivatives gives

$$\frac{d\vec{R}}{dt} = \vec{V} = \vec{v} + \vec{V}_o \dots\dots..(2)$$ ***The velocity of particle P with respect to fixed and moving system***

$$\frac{d^2\vec{R}}{dt^2} = \vec{A} = \vec{a} + \vec{A}_o \dots\dots\dots (3)$$ ***The acceleration of particle P with respect to fixed and moving system***

where: $\vec{V}_o$ and $\vec{A}_o$ are the velocity and acceleration respectively, of particle $P$ of the moving origin.

$\vec{v}$ and $\vec{a}$ are the velocity and acceleration respectively, of particle $P$ in the moving system.

$\vec{A}_o = 0$ For not accelerating moving system

$\vec{A} = \vec{a} \dots\dots\dots\dots(4)$ *When moving system is not accelerating* (*not rotating*)

المعادلة الاخيرة تبين ان التعجيل هونفسه لكل من النظام المتحرك والثابت وهي تصح في حالة كون النظام المتحرك لايدور(انعدام الحركة الدورانية في المحاور المتحركة)

## 5.3 Inertial Forces

$\vec{F} = m\vec{A} \dots\dots\dots (1)$ Newton's second law

When the equation of motion in the moving system is

$$\vec{F} = m(\vec{a} + \vec{A}_o) = m\vec{a} + m\vec{A}_o \dots\dots..(2)$$

$\vec{F} - m\vec{A}_o = m\vec{a} \dots\dots\dots\dots (3)$ ***Equation of motion in the moving system***

where: $(-m\vec{A}_o)$ called the ***inertial term***

we can write "$\vec{F}$" $= m\vec{a}$

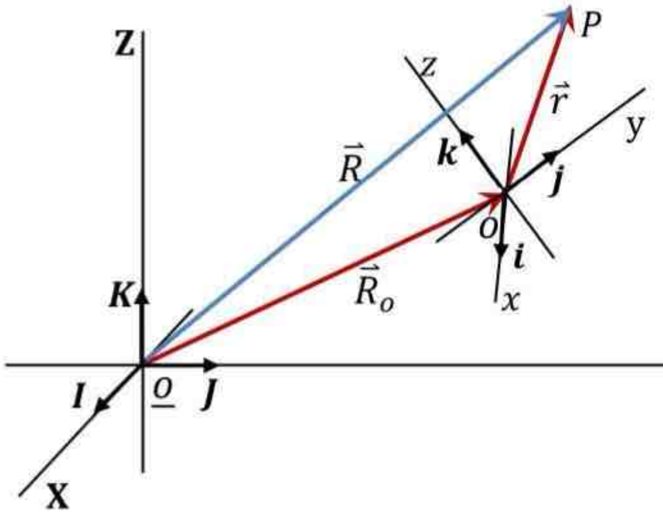
Inertial terms in the equations of motion is sometimes called *inertial forces* or *fictitious forces*. Such "forces" are not due to interactions with other physical bodies, rather, results from the acceleration of the reference system. Inertial terms are present if a noninertial coordinate system is used to describe the motion of a particle.

ان حد القصور الذاتي (Inertial terms) في معادلات الحركة يدعى أحيانا بالقوى الخاملة ، أو القوى الزائفة. مثل هذه "القوى" لا تنتج من تفاعل الجسم $P$ مع الاجسام الاخرى بأي نوع من انواع القوى الاعتيادية، بل ينتج من تسارع النظام المرجعي (reference system).

ان النظام المرجعي القصوري هو ذلك النظام الذي لا تتضمن معادلة حركته اي حد قصوري (Inertial terms)

## 5.4 General Motion of the Coordinate System

Now consider the references system undergoes both translation and rotation relative to inertial system as in figure.




$\vec{R}$: is the position vector of the particle P in the inertial system.

$\vec{r}$: is the position vector of the particle in the moving system.

$\vec{R}_0$: is the displacement of the moving origin **OO**

رض ان النظام المرجعي يتحرك حركة انتقالية ودورانية نسبة لنظام قصوري.

$$\because \vec{r} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z \qquad \dots\dots\dots (1)$$

$$\therefore \vec{R} = \vec{R}_0 + \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z \qquad \dots\dots\dots (2)$$

By differentiating with respect to the time,

$$\frac{d\vec{R}}{dt} = \frac{d\vec{R}_0}{dt} + \underbrace{\boldsymbol{i}\dot{x} + \boldsymbol{j}\dot{y} + \boldsymbol{k}\dot{z}}_{\text{Velocity of the particle relative to the moving system}} + \underbrace{x\frac{d\boldsymbol{i}}{dt} + y\frac{d\boldsymbol{j}}{dt} + z\frac{d\boldsymbol{k}}{dt}}_{\text{Velocity due to rotation of the } OXYZ \text{ coordinate}} \qquad \dots\dots\dots (3)$$

دود $\boldsymbol{i}\dot{x} + \boldsymbol{j}\dot{y} + \boldsymbol{k}\dot{z}$ تمثل سرعة الجسيم بالنسبة للمحاور المتحركة في حين ان الحدود $x\frac{d\boldsymbol{i}}{dt} + y\frac{d\boldsymbol{j}}{dt} + z\frac{d\boldsymbol{k}}{dt}$
ل سرعة الحركة الدورانية للمنظومة المتحركة $(OXYZ)$ او السرعة الناتجة من دوران المنظومة

$\dot{\vec{r}} = \boldsymbol{i}\dot{x} + \boldsymbol{j}\dot{y} + \boldsymbol{k}\dot{z}$ ……… (4) ***Velocity of the particle relative to the moving system***

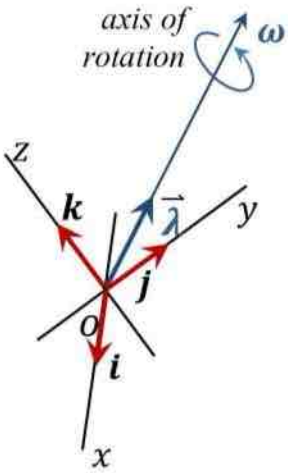
لنفرض ان اتجاه الدوران له وحدة قيمة هي $(\lambda)$ وان الانطلاق للدوران حول هذا المتجه هو $(\omega)$ فاذن $(\omega\lambda)$ هي السرعة الزاويه للمحاور الدائرة.

Let

$\lambda$: be the direction of the axis of rotation of the $OXYZ$ system.

$\omega$: be the angular speed of rotation about that axis.

As in figure




$\vec{\omega} = \omega\,\lambda$ …… (4) ***Angular velocity of the rotating coordinate system***

The direction of the angular velocity vector is given by the right-hand rule as in the definition of the cross product.

اتجاه متجه السرعة الزاوية يحدد من خلال قاعدة اليد اليمنى كما في تعريف الضرب الاتجاهي.

The time derivatives of the basis vectors ($\frac{d\boldsymbol{i}}{dt}$, $\frac{d\boldsymbol{j}}{dt}$, $\frac{d\boldsymbol{k}}{dt}$) in term of the rotation $\vec{\omega}$, we can express $\frac{d\boldsymbol{i}}{dt}$ as the cross product of $\vec{\omega}$ and $\boldsymbol{i}$

$$\frac{d\boldsymbol{i}}{dt} = \vec{\omega} \times \boldsymbol{i} \quad \ldots\ldots\ldots (11)$$

Similarly

$$\left.\begin{aligned}\frac{d\boldsymbol{j}}{dt} &= \vec{\omega} \times \boldsymbol{j}\\ \frac{d\boldsymbol{k}}{dt} &= \vec{\omega} \times \boldsymbol{k}\end{aligned}\right\} \ldots\ldots\ldots (12)$$

$$\begin{aligned}\therefore x\frac{d\boldsymbol{i}}{dt} + y\frac{d\boldsymbol{j}}{dt} + z\frac{d\boldsymbol{k}}{dt} &= x(\vec{\omega} \times \boldsymbol{i}) + y(\vec{\omega} \times \boldsymbol{j}) + z(\vec{\omega} \times \boldsymbol{k})\\ &= \vec{\omega} \times (\boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z)\\ &= \vec{\omega} \times \vec{r}\end{aligned}$$

So the velocity of particle $P$, due to rotation of the fixed coordinates system.

$$\frac{d\vec{R}}{dt} = \frac{d\vec{R}_0}{dt} + \boldsymbol{i}\dot{x} + \boldsymbol{j}\dot{y} + \boldsymbol{k}\dot{z} + \vec{\omega} \times \vec{r} \ldots\ldots\ldots (13)$$

$$\frac{d\vec{R}}{dt} = \frac{d\vec{R}_0}{dt} + \dot{\vec{\boldsymbol{r}}} + \vec{\omega} \times \vec{\boldsymbol{r}}$$

$$\vec{V} = \vec{V}_o + \dot{\vec{\boldsymbol{r}}} + \vec{\omega} \times \vec{\boldsymbol{r}} \dots \dots (14)$$

The above eqn. express the relation between the time derivatives of the position vectors of a moving particle in two coordinate systems one system regarded as fixed, the other as moving and rotating. The term $\vec{V}_o$ is due to translation of the moving system. In the case of pure rotation, it not presents.

المعادلة (14) تبين العلاقة بين المشتقات الزمنية لمتجهات الموقع لجسيم متحرك في نظامين إحداثيين، احدهما ثابتًا، والأخر يتحرك (ينتقل) ويدور. الحد $\vec{V}_0$ ظهر بسبب الحركة الانتقالية. في حالة كون الحركة دورانيه فقط فان هذا الحد لايظهر.

For any vector $\vec{\boldsymbol{q}}$

$$\frac{d\vec{\boldsymbol{q}}}{dt} = \dot{\vec{\boldsymbol{q}}} + \vec{\omega} \times \vec{\boldsymbol{q}}$$

where:

$\dot{\boldsymbol{q}}$ is the time rate of change of $\vec{\boldsymbol{q}}$ in the rotating system $(\boldsymbol{i}\dot{q}_x + \boldsymbol{j}\dot{q}_y + \boldsymbol{k}\dot{q}_z)$

$\vec{\omega} \times \vec{\boldsymbol{q}}$ is the time rate of change of $\vec{\boldsymbol{q}}$ arising from rotation of coordinate system $(q_x \frac{di}{dt} + q_y \frac{dj}{dt} + q_z \frac{dk}{dt})$

Let find the relation between the ***acceleration vectors***, from eqn. (14)

بالامكان ايجاد العلاقة بين متجهات التعجيل في حالة الحركة الانتقالية والدورانية للمحاور الاحداثية، من المعادلة (14)

$$\frac{d\vec{R}}{dt} = \vec{V} = \vec{V}_o + \dot{\vec{\boldsymbol{r}}} + \vec{\omega} \times \vec{\boldsymbol{r}}$$

$$\frac{d\vec{R}}{dt} = \vec{V}_o + (\,\boldsymbol{i}\dot{x} + \boldsymbol{j}\dot{y} + \boldsymbol{k}\dot{z}) + (\vec{\omega} \times \vec{r})$$

$$\frac{d^2\vec{R}}{dt^2} = \dot{\vec{V}}_o + \frac{d}{dt}(\boldsymbol{i}\dot{x} + \boldsymbol{j}\dot{y} + \boldsymbol{k}\dot{z}) + \frac{d}{dt}(\vec{\omega} \times \vec{r})$$

$$\vec{A} = \vec{A}_o + \frac{d}{dt}(\,\boldsymbol{i}\dot{x} + \boldsymbol{j}\dot{y} + \boldsymbol{k}\dot{z}) + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

$$= \vec{A}_o + \boldsymbol{i}\ddot{x} + \dot{x}\frac{di}{dt} + \boldsymbol{j}\ddot{y} + \dot{y}\,\frac{dj}{dt} + \boldsymbol{k}\ddot{z} + \dot{z}\,\frac{dk}{dt} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \Big[\,\boldsymbol{i}\dot{x} + \boldsymbol{j}y + \boldsymbol{k}\dot{z} + x\frac{di}{dt} + y\frac{dj}{dt} + z\frac{dk}{dt}\Big]$$

$$= \vec{A}_o + \boldsymbol{i}\,\ddot{x} + \boldsymbol{j}\ddot{y} + \boldsymbol{k}\ddot{z} + \dot{x}\,\frac{di}{dt} + \dot{y}\,\frac{dj}{dt} + \dot{z}\,\frac{dk}{dt} + \vec{\dot{\omega}} \times \vec{r} + \vec{\omega} \times \left[\,\vec{\dot{r}} + x(\vec{\omega} \times \boldsymbol{i}) + y(\vec{\omega} \times \boldsymbol{j}) + z(\vec{\omega} \times \boldsymbol{k})\right]$$

$$= \vec{A}_o + \vec{\ddot{r}} + \dot{x}(\vec{\omega} \times \boldsymbol{i}) + \dot{y}(\vec{\omega} \times \boldsymbol{j}) + \dot{z}(\vec{\omega} \times \boldsymbol{k}) + \vec{\dot{\omega}} \times \vec{r} + \vec{\omega} \times [\,\dot{r} + \vec{\omega} \times (\boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z)]$$

$$= \vec{A}_0 + \vec{\ddot{r}} + \vec{\omega} \times (\,\boldsymbol{i}\dot{x} + \boldsymbol{j}\dot{y} + \boldsymbol{k}\dot{z}) + \vec{\dot{\omega}} \times \vec{r} + \vec{\omega} \times \left[\vec{\dot{r}} + (\vec{\omega} \times \vec{r})\right]$$

$$= \vec{A}_0 + \vec{\ddot{r}} + \vec{\omega} \times \vec{\dot{r}} + \vec{\dot{\omega}} \times \vec{r} + \vec{\omega} \times \vec{\dot{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\vec{A} = \vec{A}_o + \vec{\ddot{r}} + 2\vec{\omega} \times \vec{\dot{r}} + \vec{\dot{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \;\ldots\ldots (15)$$

Eqn.(15) giving the acceleration in the fixed system in terms of the position, velocity, and acceleration in the rotating system.

$\vec{A}_o$: ***Acceleration of the moving system***

$\vec{\ddot{r}}$: ***Acceleration of particle in the rotating system***

$2\vec{\omega} \times \dot{r}$: ***Coriolis acceleration***

$\vec{\dot{\omega}} \times r$: ***Transverse acceleration***

$\vec{\omega} \times (\vec{\omega} \times \vec{r})$: ***Centrifugal acceleration***

المعادلة (15) تمثل التعجيل في النظام الثابت من حيث الموقع والسرعة والتعجيل في النظام المتحرك (يدور).

$\vec{A}_o$: تعجيل النظام المتحرك

$\vec{\ddot{r}}$: تعجيل الجسيمة في النظام الدوراني

$2\vec{\omega} \times \dot{r}$ : تعجيل كوريولس

$\vec{\dot{\omega}} \times r$ : التعجيل المستعرض

$\vec{\omega} \times (\vec{\omega} \times \vec{r})$ : تعجيل القوة الطاردة المركزيه (الجذب المركزي) وهي تتجه دائماً نحو محور الدوران وتكون عمودي على المحور.

**Example:**

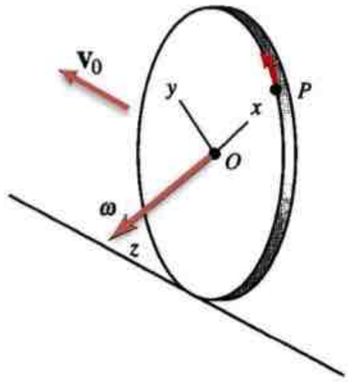
A wheel of radius ***b*** rolls along the ground with constant forward speed $V_o$. Find the acceleration, relative to the ground, of any point on the rim.

عجلة نصف قطرها $\boldsymbol{b}$ تتدحرج على سطح الارض بسرعة ثابتة $V_o$. جد التعجيل لأية نقطه على محيط العجلة بالنسبة للارض لاية نقطة على الحافة.

**Solution:**

Let us choose a coordinate system fixed to the rotating wheel, and let the moving origin be at the center with the *x*-axis passing through the point in question,

يتضح من السؤال ان العجلة تمثل حالة نظام احداثيات يمتلك سرعة خطية وسرعة دورانية (يتحرك ويدور) في حين انها تتحرك على سطح ثابت هو سطح الارض. اي ان هناك نظامين احداثيين المتحرك فقط والنظام الاخر يتحرك ويدور. لحل السؤال يجب اولا تحديد نظام احداثيات ثابت للعجلة الدائرة بحيث يكون مركز العجلة فيه هو نقطة الاصل والذي يمتلك سرعة الخطية $V_o$.




Position vector is: $\vec{r} = \boldsymbol{i}b$

Velocity vector is: $\dot{\vec{r}} = 0$

Acceleration vector is: $\ddot{\vec{r}} = 0$

The angular velocity vector is given by:

$$\vec{\omega} = \boldsymbol{k}\omega = \boldsymbol{k}\frac{V_o}{b} \qquad \text{السرعة الزاوية} = \frac{\text{الانطلاق الخطي}}{\text{نصف القطر}}$$

for the choice of coordinates shown; therefore, all terms in the expression for acceleration vanish except the centripetal term:

حدود التعجيل في معادلة التعجيل تتلاشى ماعدا حد الجذب المركزي.

$V_o$ انطلاق ثابت $\leftarrow$ $\dot{\vec{\omega}} = 0$

$$\vec{A} = \ddot{\vec{r}} + 2\vec{\omega}\times\dot{\vec{r}} + \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \vec{A}_o$$

$$= 0 + 0 + 0 + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + 0$$

$$= \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$= \boldsymbol{k}\omega\times(\boldsymbol{k}\omega\times\boldsymbol{i}b)$$

$$= \boldsymbol{k}\frac{V_o}{b}\times\left(\boldsymbol{k}\frac{V_o}{b}\times\boldsymbol{i}b\right) = \boldsymbol{k}\frac{{V_o}^2}{b}\times(\boldsymbol{k}\times\boldsymbol{i})$$

$= \frac{V_o^2}{b} \boldsymbol{k} \times \boldsymbol{j} = \frac{V_o^2}{b}(-\boldsymbol{i})$

$\therefore A = -\boldsymbol{i}\frac{V_o^2}{b}$ is always directed toward the center of the rolling wheel.

يكون دائماً باتجاه مركز العجلة الدائرة.

**Example:**

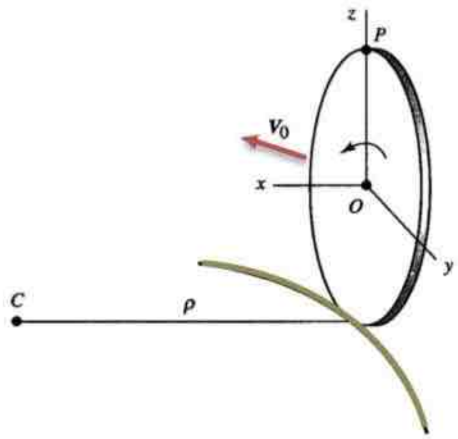
A bicycle travels with constant speed $V_o$ around a track of radius $\rho$. What is the acceleration of the highest point on one of its wheels?

دراجه هوائيه تسير بانطلاق ثابت على طريق منحني نصف قطره $\rho$. ما التعجيل في اعلى نقطه لاي من عجلات الدراجة؟

**Solution:**

Choose a coordinate system with origin at the center of the wheel and with the $\boldsymbol{x}$-axis horizontal pointing toward the center of curvature $C$ of the track. Rather than have the moving coordinate system rotate with the wheel, choose a system in which the $\boldsymbol{z}$-axis remains vertical, the $\boldsymbol{oxyz}$ system rotates

ان الدراجة في حالة حركة انتقالية بسرعة $V_o$ وعجلاتها تدور حول مسار دائري نصف قطره $\rho$. نختار نظام احداثيات نقطة الاصل له تكون في مركز عجلة الدراجة، بحيث يكون محور $\boldsymbol{x}$ افقي ويؤثر باتجاه (نحو) مركز تكور الطريق ($C$). والمحور العمودي $\boldsymbol{z}$ يبقى عموديا حتى عند انحناء العجلة. عليه فان النظام ($oxyz$) يدور بسرعة زاويه مقدارها ω




angular velocity express as:

$\vec{\omega} = \boldsymbol{k}\frac{V_o}{\rho}$

Acceleration of the moving origin $\vec{A}_o$ is given by:

$\vec{A}_o = \boldsymbol{i}\frac{V_o^2}{\rho}$

تعجيل المركزي = $\frac{\text{مربع الانطلاق}}{\text{نصف القطر}}$

each point on the wheel is moving in a circle of radius $\boldsymbol{b}$ with respect to the moving origin, the ***acceleration*** in the ($oxyz$ ) system of any point on the wheel is directed toward (o) and has magnitude $\frac{V_o{}^2}{b}$. Thus, in the moving system we have

كل نقطة على العجلة تتحرك بدائرة نصف قطرها $b$ (نصف قطر العجلة) بالنسبة لنقطه الاصل المتحركة فالتعجيل في المنظومة $(oxyz)$ لأية نقطة على العجله تتجه نحو $(o)$ وقيمته $(\frac{V_o{}^2}{b})$

$$\ddot{\vec{r}} = -\boldsymbol{k}\frac{V_o{}^2}{b}$$

for the point at the ***top*** of the wheel. Also, the ***velocity*** of this point in the moving system is given by:

في أعلى نقطة على العجلة

$$\dot{\vec{r}} = -\boldsymbol{j}V_o$$

so the Coriolis acceleration is

$$2\vec{\omega} \times \dot{\vec{r}} = 2\left(\frac{V_o}{\rho}\boldsymbol{k}\right) \times (-\boldsymbol{j}V_o)$$

$$= 2\frac{V_o{}^2}{\rho}\boldsymbol{i}$$

Because the angular velocity $\vec{\omega}$ is constant, the transverse acceleration is zero. The centripetal acceleration is also zero because

$$\dot{\vec{\omega}} \times \vec{r} = 0$$

السرعة الزاويه $\vec{\omega}$ ثابته (حيث $V_o$ ثابت) فان $\dot{\vec{\omega}} = 0$ فان التعجيل المستعرض يكون صفراً .

وكذلك تعجيل الجذب المركزي = صفر

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \frac{V_o}{\rho}\boldsymbol{k} \times \left(\frac{V_o}{\rho}\boldsymbol{k} \times b\boldsymbol{k}\right) = 0 \qquad (\boldsymbol{k} \times \boldsymbol{k}) = 0$$

The net acceleration, relative to the ground, of the highest point on the wheel is

التعجيل الكلي في اعلى نقطة على العجلة

$$\vec{A} = \ddot{\vec{r}} + 2\vec{\omega} \times \dot{\vec{r}} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \vec{A}_o$$

$$= -\boldsymbol{k}\frac{V_o{}^2}{b} + 2\frac{V_o{}^2}{\rho}\boldsymbol{i} + 0 + 0 + \frac{V_o{}^2}{\rho}\boldsymbol{i}$$

$$\vec{A} = 3\,\frac{V_o{}^2}{\rho}\boldsymbol{i} - \frac{V_o{}^2}{b}\boldsymbol{k}$$

## 5.5 Dynamics of a Particle in a Rotating Coordinate System

The fundamental equation of motion of a particle in an inertial frame of reference is

معادلة الحركة جسيم في احداثيات القصور الذاتي

$\vec{F} = m\frac{d^2\vec{R}}{dt^2}$ ... ... (1)

$\vec{F}$: is the physical force acting on the particle

$\frac{d^2\vec{R}}{dt^2} = \vec{A} = \ddot{\vec{r}} + 2\vec{\omega} \times \dot{\vec{r}} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \vec{A}_o$

The equation of motion in a noninertial frame of reference as

$\therefore F - m\vec{A}_o - 2m\,\vec{\omega} \times \dot{\vec{r}} - m\dot{\vec{\omega}} \times \vec{r} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = m\ddot{\vec{r}}$ ... ... (2)

Eqn. (2) represents the dynamical equation of motion of a particle in a noninertial frame of reference subjected to both real, physical forces as well as those inertial forces that appear as a result of the acceleration of the noninertial frame of reference.

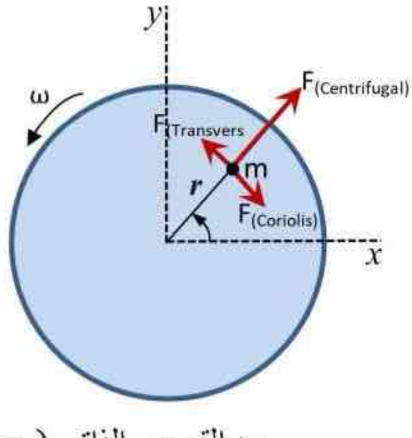
معادلة (2) تمثل معادلة الحركة لجسيم في احداثيات غير قصورية يخضع لقوى فعلية وفيزيائية اضافة الى قوى القصور الذاتي التي تظهر كنتيجة لتسارع الإطار المرجعي غير القصوري.

$\vec{F} = (\boldsymbol{Physical\ force})$ قوة فيزيائية

$\vec{F}_{cor} = -2m\vec{\omega} \times \dot{\vec{r}}$ $(\boldsymbol{Coriolis\ force})$ قوة كوريولس

$\vec{F}_{trans} = -m\dot{\vec{\omega}} \times \vec{r}$ $(\boldsymbol{Transverse\ force})$ القوة المستعرضة

$\vec{F}_{cent} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})$ $(\boldsymbol{Centrifugal\ force})$ القوة الطاردة المركزية




$-mA_0$ (***Inertial term*** due to translational of coordinate system ) حد القصور الذاتي

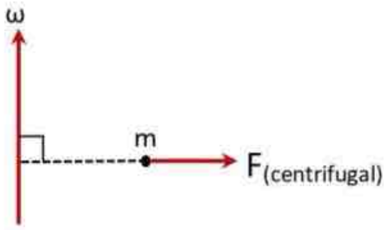
$-mA_0$ تمثل القوة الناتجه من دوران منظومة المحاور (باتجاه منظومة الاحداثيات المنتقلة).

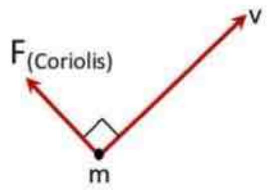
قوى القصور الذاتي ناتجه من الصفات الذاتيه للمادة اكثر من كونها ناتجه من تأثير الاجسام الاخرى.

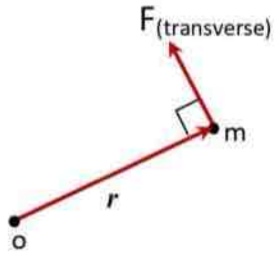
"F" $= m\ddot{\vec{r}} = m(\boldsymbol{i}\ddot{x} + \boldsymbol{j}\ddot{y} + \boldsymbol{k}\ddot{z})$ ... ... . ... (3)

where

"F" $= \vec{F}_{cor} + \vec{F}_{trans} + \vec{F}_{cent} - mA_o$ ... . . ... (4)








**1.** The ***Coriolis force*** is particularly interesting. It is present only if a particle is moving in a rotating coordinate system. Its direction is always perpendicular to the velocity vector of the particle in the moving system. The Coriolis force thus seems to deflect a moving particle at right angles to its direction of motion.

- This force is important in computing the trajectory of a projectile.
- Coriolis effects are responsible for the circulation of air around high- or low-pressure systems on Earth's surface. In the case of a high-pressure area, as air spills down from the high, it flows outward and away, deflecting toward the right as it moves into the surrounding low, setting up a clockwise circulation pattern. In the Southern Hemisphere the reverse is true.

**2.** The ***transverse force*** is present only if there is an angular acceleration (or deceleration) of the rotating coordinate system. This force is always perpendicular to the radius vector ***r*** in the rotating coordinate system.

**3.** The ***centrifugal force*** is the familiar one that arises from rotation about an axis. It is directed outward away from the axis of rotation and is perpendicular to that axis.

1. قوة كوريولس تظهر فقط عندما يتحرك الجسم تحت الشروط الدورانيه للمنظومة وتتجه دائماً بشكل عمودي على متجه السرعة للجسم في نظام الاحداثيات المتحرك. لذلك فان هذه القوة تميل الى جعل الجسم المتحرك يحيد عن مساره بزوايا عمودية على مساره.

من التطبيقات المهمة لهذه القوة:

- حساب مسار القذيفة trajectory of projectile
- تأثيرات كوريولس المسؤولة عن دوران الرياح في المناطق ذات الضغط العالي والواطيء على سطح الارض ففي حالة مناطق الضغط العالي تحاول الرياح ان تجري بأتجاه اليمين في نصف الكرة الجنوبي والعكس صحيح.

2. القوة المستعرضة تظهر فقط عندما يتواجد تعجيل زاوي لحركة منظومة الاحداثيات الدورانيه وهذه القوة تكون ايضاً عموديه على قيمه نصف القطر $(\vec{r})$.

3. القوه الطاردة تنشأ من الدوران حول محور وهذه القوه تتجه دائماً بعيداً عن محور الدوران وتكون عمودية عليه.

## Example:

A bug moving out ward with constant speed $u$ along the spoke of a wheel, which is rotating with constant angular velocity ω about a vertical axis, find all the forces acting on the bug. Then find how far the bug can crawl before it starts to slip, given the coefficient of friction $\mu$ between the bug and the spoke.

حشرة (بعوضة) تتحرك (تزحف) بانطلاق ثابت $u$ على شعاع (سلك) عجله تدور بسرعة زاويه ثابته $\omega$ حول محور عمودي. جد كل القوى التي تؤثر عليها. ثم احسب بعد الحشرة قبل ان تبدأ بالانزلاق علماً ان معامل الاحتكاك بين الحشرة والشعاع هو μ .

**Solution**

First, let us choose a coordinate system fixed on the wheel, and let the $x$-axis point along the spoke in question.

نختار منظومة احداثيات ثابته على العجله ونفرض المحور $(x)$ يتجه على طول اتجاه حركة الجسيم .

$$u = \frac{x}{t} \Rightarrow x = ut$$

$$\therefore \vec{r} = \boldsymbol{i}x = \boldsymbol{i}ut$$

$$\dot{\vec{r}} = \boldsymbol{i}\dot{x} = \boldsymbol{i}u$$

$$\because\ u = constant \ \Rightarrow\ \ddot{\vec{r}} = 0$$

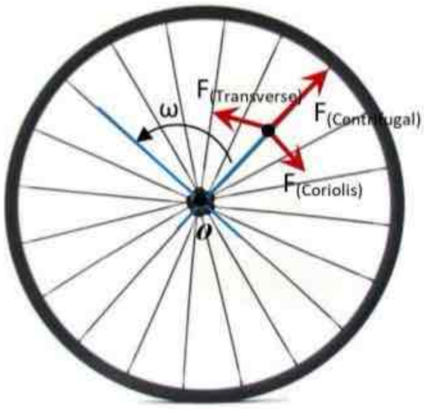
for the velocity and acceleration of the bug as described in the rotating system.

If we choose the $z$-axis to be vertical, then

اذا اخترنا المحور $(z)$ شاقوليا (عمودي)

$$\vec{\omega} = \boldsymbol{k}\omega \ \Rightarrow \dot{\vec{\omega}} = 0$$

The various forces are then given by the following:




$F_{cor} = -2m\vec{\omega} \times \dot{\vec{r}}$

$= -2m(\boldsymbol{k}\omega) \times (\boldsymbol{i}u) = -2m\omega u(\boldsymbol{k} \times \boldsymbol{i})$

$F_{cor} = -2m\omega u\boldsymbol{j}$

$F_{tran} = -m\dot{\vec{\omega}} \times \vec{r} = 0$

$F_{cent} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})$

$= -m(\boldsymbol{k}\omega) \times [(\boldsymbol{k}\omega) \times (\boldsymbol{i}x)]$

$= -m\omega^2[\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{i}x)]$

$= -m\omega^2 \, [\boldsymbol{k} \times \boldsymbol{j}x]$

$= m\omega^2 x\boldsymbol{i}$

$\therefore \vec{F} - m\vec{A}_0 - 2m\vec{\omega} \times \dot{\vec{r}} - m\dot{\vec{\omega}} \times \vec{r} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = m\ddot{\vec{r}}$

$\vec{F} - 0 - 2m\omega u\boldsymbol{j} - 0 + m\omega^2 x \, \boldsymbol{i} = 0$

$F = 2m\omega u\boldsymbol{j} - m\omega^2 x\boldsymbol{i}$

Here $F$ is the real force exerted on the bug by the spoke

$F$ تمثل القوة الحقيقية المؤثرة على الحشرة من قبل سلك العجلة.

Because the force of friction $\vec{F}$ has a maximum value of slipping starts when

ان قوة الاحتكاك تكون في قيمتها العظمى عندما $\vec{F} = \mu mg$

$|\vec{F}| = \mu mg = [(2m\omega u)^2 + (m\omega^2 \, x)^2]^{1/2}$

$\mu^2 \, m^2 g^2 = (2m\omega u)^2 + (m\omega^2 x)^2$

$(m\omega^2 x)^2 = \mu^2 m^2 g^2 - (2m\omega u)^2$

$m^2\omega^4 x^2 = \mu^2 m^2 g^2 - 4m^2\omega^2 u^2$

$m^2\omega^4 x^2 = m^2(\mu^2 g^2 - 4\omega^2 u^2)$

$x^2 = \frac{m^2(\mu^2 g^2 - 4\omega^2 u^2)}{m^2 \, \omega^4}$

$x = \frac{(\mu^2 g^2 - 4\omega^2 u^2)^{1/2}}{\omega^2}$ The distance the bug can crawl before slipping

وهي المسافة التي تقطعها الحشرة قبل ان تبدأ بالانزلاق.