

بسم الله الرحمن الرحيم

Subject

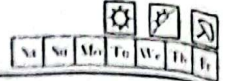
Year

Month

Date

نظرية الحلقات

Ring theory



Def:- The triple $(R, *, \circ)$ is said to be ring

شروط الحلقة

1. $(R, *)$ is Comm. group.

a. $\forall a, b \in R \rightarrow a * b \in R$ is closed

b. $\forall a, b, c \in R \rightarrow a * (b * c) = (a * b) * c \therefore \text{ass}$

c. $\exists e \in R \rightarrow a * e = e * a = a \rightarrow \forall a \in R \therefore \text{id}$

d. $\exists a' \in R \rightarrow a' * a = a * a' = e \rightarrow \forall a \in R \therefore \text{inv}$

f. $\forall a, b \in R \rightarrow a * b = b * a \therefore \text{comm}$

2. $(R, \circ)$ is semi group.

a. $\forall a, b \in R \rightarrow a \circ b \in R$ is closed

b. $\forall a, b, c \in R \rightarrow (a \circ b) \circ c = a \circ (b \circ c) \therefore \text{ass}$

3. $(\circ)$ dist $(*)$

a. $a \circ (b * c) = (a \circ b) * (a \circ c)$

b. $(b * c) \circ a = (b \circ a) * (c \circ a)$

Example:- which of the following math. system is ring? $(\mathbb{Z}, +, \cdot)$

Sol 1. $(\mathbb{Z}, +)$ is comm. group. 2. $(\mathbb{Z}, \cdot)$ is semi group.

3. $(\cdot)$ dist $(+)$

$\forall a, b, c \in \mathbb{Z} \quad 1, 2, 3 \in \mathbb{Z}$

$a(b+c) = (ab) + (ac)$

1. $(2+3) = (1 \cdot 2) + (1 \cdot 3) \rightarrow 5 = 5$

$\therefore (\mathbb{Z}, +, \cdot)$ is ring

H. w - which of the following math. sys. is ring?

1. $(\mathbb{R}, +, \cdot)$ 2. $(\mathbb{Q}, +, \cdot)$ 3. $(\mathbb{C}, +, \cdot)$

Sol 1. $(\mathbb{R}, +, \cdot)$

$$R = \{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{I}\mathbb{Q}\}$$

$$\mathbb{Q} \cap \mathbb{I}\mathbb{Q} = \emptyset$$

1. $(\mathbb{R}, +)$ is Comm. group

a. $\forall a, b \in \mathbb{R} \rightarrow a + b \in \mathbb{R}$ is closed

b. $\forall a, b, c \in \mathbb{R} \rightarrow (a + b) + c = a + (b + c)$

$$1 + (2 + 3) = 1 + (2 + 3)$$

$$6 = 6 \quad \therefore \text{is ass}$$

c. $e = 0 \in \mathbb{R} \rightarrow a + 0 = 0 + a = a \rightarrow a \in \mathbb{R}$

$$2 + 0 = 0 + 2 = 2$$

d. $\forall a \in \mathbb{R} \rightarrow a + a' = a' + a = e \rightarrow \exists e = 0$

$$2 + (-2) = (-2) + 2 = 0$$

f. $\forall a, b \in \mathbb{R} \rightarrow a + b = b + a$

$$2 + 3 = 3 + 2$$

$$5 = 5 \quad \therefore \text{is comm.}$$

2. $(\mathbb{R}, \cdot)$ is Semi group.

a. $\forall a, b \in \mathbb{R} \rightarrow a \cdot b \in \mathbb{R}$ ^{closed} $\therefore$ is closed

b. $\forall a, b, c \in \mathbb{R} \rightarrow a \cdot (b \cdot c) = (a \cdot b) \cdot c$ is ass

$$1 \cdot (2 \cdot 3) = (1 \cdot 2) \cdot 3$$

$$6 = 6$$

3. $(\cdot)$ dist $(+)$

$\forall a, b, c \in \mathbb{R} \rightarrow a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

$$1 \cdot (2 + 3) = (1 \cdot 2) + (1 \cdot 3)$$

$$1 \cdot 5 = 2 + 3$$

$$5 = 5$$

SHAFAGH $(\mathbb{R}, +, \cdot)$ is ring

2. $(\mathbb{Q}, +, \cdot)$

اعداد نسبية
H.W. حل $\mathbb{Q} = \{ \frac{a}{b} \mid a \text{ عدده طبيعي } b \neq 0 \}$

Sol 1. $(\mathbb{Q}, +)$ is Comm. group.

2. $(\mathbb{Q}, +)$ is semi group.

3. $(\cdot)$ dist $(+)$

$$\forall a, b, c \in \mathbb{Q}$$

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

$$\frac{1}{2} \cdot \left(\frac{1}{3} + \frac{2}{3} \right) = \left(\frac{1}{2} \cdot \frac{1}{3} \right) + \left(\frac{1}{2} \cdot \frac{2}{3} \right)$$

$$\frac{1}{2} \cdot \frac{3}{3} = \frac{1}{6} + \frac{2}{6}$$

$$\rightarrow \frac{3}{6} = \frac{3}{6} \rightarrow \frac{1}{2} = \frac{1}{2}$$

$(\mathbb{Q}, +, \cdot)$ is ring

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3. $(\mathbb{C}, +, \cdot)$

$\mathbb{C} = a + bi$ أعداد مركبة

Sol 1. $(\mathbb{C}, +)$ is Comm. group.

2. $(\mathbb{C}, \cdot)$ is semi group.

20 3. $(\cdot)$ dist $(+)$

$$\forall (a_1 + bi_1), (a_2 + bi_2), (a_3 + bi_3) \in \mathbb{C}$$

$$(a_1 + bi_1) \cdot [(a_2 + bi_2) + (a_3 + bi_3)]$$

الطرف الرئيس

$$[(a_1 + bi_1) \cdot (a_2 + bi_2)] + [(a_1 + bi_1) \cdot (a_3 + bi_3)]$$

الطرف اليمين

$\therefore (\mathbb{C}, +, \cdot)$ is ring.

حلقة ابدالية

Def:- A ring $(R, *, +)$ is said to be comm. ring if $a * b = b * a \in R$.

* تكون الحلقة ابدالية اذا كانت الخرب ابدالية بعملية ثانية

حلقة ذات عنصر محايد

Def:- A ring $(R, *, +)$ is said to have identity element if there exists $e \in R$ s.t $a * e = e * a = a \forall a \in R$

* تكون الحلقة عنصر محايد اذا كانت الخرب عنصر محايد

Comm. ring with identity

حلقة ابدالية ذات عنصر محايد

* تكون حلقة ابدالية ذات عنصر محايد اذا حققت الشروط

Example :- $(\phi, +, \cdot)$ is a comm. ring with identity
 حلقة ابالية ذات عنصر محايد

Sol. 1. $(\phi, +)$ is comm. group.

$$a, (a_1 + bi_1), (a_2 + bi_2) \in \phi \Rightarrow (a_1 + bi_1) + (a_2 + bi_2) \in \phi$$

$$b, (a_1 + bi_1), (a_2 + bi_2), (a_3 + bi_3) \in \phi$$

$$\Rightarrow (a_1 + bi_1) + [(a_2 + bi_2) + (a_3 + bi_3)] = [(a_1 + bi_1) + (a_2 + bi_2)] + (a_3 + bi_3)$$

$$c, (a + bi) \in \phi, \exists e = 0 + 0i \in \phi$$

$$\Rightarrow (a + bi) + (0 + 0i) = (a + bi) \in \phi$$

$$d, (a + bi) \in \phi, \exists e = 0 + 0i \in \phi$$

$$\Rightarrow (a + bi) + (-a - bi) = 0 + 0i \in \phi$$

$$f, (a_1 + bi_1), (a_2 + bi_2) \in \phi$$

$$\Rightarrow (a_1 + bi_1) + (a_2 + bi_2) = (a_2 + bi_2) + (a_1 + bi_1)$$

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2. $(\phi, \cdot)$ is semi group.

$$a, \forall (a_1 + bi_1), (a_2 + bi_2) \in \phi$$

$$(a_1 + bi_1) \cdot (a_2 + bi_2) \in \phi$$

$$b, \forall (a_1 + bi_1), (a_2 + bi_2), (a_3 + bi_3) \in \phi$$

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$$(a_1 + bi_1) [(a_2 + bi_2) (a_3 + bi_3)] = [(a_1 + bi_1) (a_2 + bi_2)] (a_3 + bi_3)$$

3. $(\cdot)$ dist $(+)$

$$\forall (a_1 + bi_1), (a_2 + bi_2), (a_3 + bi_3) \in \phi$$

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$$(a_1 + bi_1) [(a_2 + bi_2) + (a_3 + bi_3)]$$

الطرف اليمين

$$[(a_1 + bi_1) (a_2 + bi_2)] + [(a_1 + bi_1) (a_3 + bi_3)]$$

الطرف اليمين

SHAFAGH

is ring



يتبع

$(\phi, \cdot)$ is Comm. ring

$$\forall (a_1 + bi_1), (a_2 + bi_2) \in \phi$$

$$(a_1 + bi_1)(a_2 + bi_2) = (a_2 + bi_2)(a_1 + bi_1)$$

اباليت بالقرين

حسب ملاحظة

$(\phi, \cdot)$ $\exists$ identity

$$\forall (a + bi) \in \phi \quad \exists e = 1 + 0i \in \phi$$

$$(a + bi)(1 + 0i) = (a + bi)$$

$\therefore (\phi, +, \cdot)$ is Comm. ring with identity.

Example:- $(\mathbb{Z}, +, \cdot)$ is Comm. ring with identity?

حلقة اباليت ذات عنصر واحد

sol 1, $(\mathbb{Z}, +)$ is Comm. group.

2, $(\mathbb{Z}, \cdot)$ is Semi group.

3, $(\cdot)$ dist $(+)$

$$\forall a, b, c \in \mathbb{Z}$$

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

4, $(\mathbb{Z}, \cdot)$ Comm. ring.

$$a, b \in \mathbb{Z}$$

$$a \cdot b = b \cdot a$$

5, $(\mathbb{Z}, \cdot)$ $\exists$ identity

$$a \in \mathbb{Z}, \exists 1 \in \mathbb{Z}$$

$$a \cdot 1 = a$$

$$a \cdot 1 = a$$

$\therefore (\mathbb{Z}, +, \cdot)$ is Comm. ring with identity

دقيقة ثانية من الرسالة - (فرافات) (أو x)

Q/give example of Comm. ring?

لهم مثال عن حلقة ابدالية.

Sol :- $(\mathbb{Z}, +, \cdot)$ is Comm. ring.

Q/give example of Comm. ring with identity?

لهم مثال عن حلقة ابدالية ذات عنصر محايد

Sol :- $(\mathbb{Z}, +, \cdot)$ is comm. ring with identity.

10 $(\mathbb{Z}, \cdot) \quad e = 1$

$$a \cdot e = a \rightarrow a \cdot 1 = a$$

$(\mathbb{Z}, \cdot)$ is comm.

$$\forall a, b \in \mathbb{Z} \quad a \cdot b = b \cdot a$$

$$2 \cdot 3 = 3 \cdot 2$$

$$6 = 6$$

15 Q/give example of Comm. ring without ident?

Sol :- $(\mathbb{Z}_e, +, \cdot)$

بالعناصر المحايه للفريه هو ا

$(\mathbb{Z}, \cdot) \quad \text{idem}(1) \notin \mathbb{Z}_e$

$\mathbb{Z}_e \neq 1$ ليس في $\mathbb{Z}$ اعداد موجبة

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H.W :- $(\mathbb{Z}_0, +, \cdot)$ is Comm. ring with identity?

Sol :- not Comm. ring with identity.

$$(\mathbb{Z}_0, +) \rightarrow \text{idem}(0) \notin \mathbb{Z}_0$$

بالعناصر المحايه للجمع

واله $0 \in \mathbb{Z}_0$ $0 \notin \mathbb{Z}_0$

لن $0 \in \mathbb{Z}$ اعداد سالبة.

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$$(M_{2 \times 2}, +, \cdot) \rightarrow$$

حلقة ليست

ابدالية لكن تحتويه

Sol :- is ring nat

عنصر محايد

Comm. ring

SHAFAGH is ring with identity.

Example: prove that $(\mathbb{Z}_4, +, \cdot)$ is ring?

Sol: $\mathbb{Z}_4 = \{0, 1, 2, 3\}$

1. $(\mathbb{Z}_4, +_4)$ is comm. group.

a. from the table R is

Closed $+_4$

b. $\forall a, b, c \in \mathbb{Z}_4$

$$(a +_4 b) +_4 c = a +_4 (b +_4 c)$$

$$(2 +_4 2) +_4 3 = 2 +_4 (2 +_4 3)$$

$$0 +_4 3 = 2 +_4 1$$

$$3 = 3$$

c. $e = 0$

$$d. a + a^{-1} = e \rightarrow a + a^{-1} = 0 \rightarrow 3 +_4 1 = 0$$

$$e. a +_4 b = b +_4 a \rightarrow 2 +_4 3 = 3 +_4 2 \rightarrow 1 = 1$$

$\forall a, b \in \mathbb{Z}_4$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

2. $(\mathbb{Z}_4, \cdot_4)$ is semi group.

a. From the table R is

Closed $\cdot_4$

$$b. \forall a, b, c \in \mathbb{Z}_4 \rightarrow a \cdot_4 (b \cdot_4 c) = (a \cdot_4 b) \cdot_4 c$$

$$\rightarrow 1 \cdot_4 (2 \cdot_4 3) = (1 \cdot_4 2) \cdot_4 3$$

$$\rightarrow 2 = 2$$

3. $(\cdot_4)$ dist $(+_4)$

$$a \cdot_4 (b +_4 c) = (a \cdot_4 b) +_4 (a \cdot_4 c)$$

$$1 \cdot_4 (2 +_4 3) = (1 \cdot_4 2) +_4 (1 \cdot_4 3)$$

$$1 \cdot_4 1 = 2 +_4 3$$

$$1 = 1$$

$\cdot_4$	1	2	3
1	1	2	3
2	2	0	2
3	3	2	1

SHAFAGH $(\mathbb{Z}, +, \cdot)$ is ring.

Q6/ $a * b = a + b - 1$

$a \circ b = a + b - ab$

prove $(\mathbb{Z}, *, \circ)$ is a comm. ring.

Sol 1, $(\mathbb{Z}, *)$ is Comm. group.

$a, (*)$ is closed From Def. $a * b \in \mathbb{Z}$

$b, \forall a, b, c \in \mathbb{Z}$

$(a * b) * c = a * (b * c)$

$(a + b - 1) * c = a * (b + c - 1)$

$a + b - 1 + c - 1 = a + b + c - 1 - 1$

$a + b + c - 2 = a + b + c - 2$

$c, e = ?$

$a * e = e * a = a$

$-1 + a + e = a \Rightarrow$

$e - 1 = a - a \Rightarrow e - 1 = 0 \Rightarrow e = 1$

$d, a^{-1} = ?$

$a * a^{-1} = a^{-1} * a = e$

$a + a^{-1} - 1 = e \Rightarrow a + a^{-1} - 1 = 1$

$\Rightarrow a + a^{-1} = 2 \Rightarrow a^{-1} = 2 - a$

20 $f, a * b = b * a$

$a + b - 1 = b + a - 1$

2, $(\mathbb{Z}, \circ)$ is semi group.

$a, \forall a, b \in \mathbb{Z}, a \circ b \in \mathbb{Z}$ is closed

$b, \forall a, b, c \in \mathbb{Z}, a \circ (b \circ c) = (a \circ b) \circ c$

$\Rightarrow (a + b + c - bc) - (ab + ac - abc) = (a + b - ab + c) - (ac + bc - abc)$

$\Rightarrow a + b + c - bc - ab - ac + abc = a + b - ab + c - ac - bc + abc$

3. $(\mathbb{Z}, *)$ dist. $(*)$

$$a \circ (b * c) = (a \circ b) * (a \circ c)$$

$$a \circ (b + c - 1) = (a + b - ab) * (a + c - ac)$$

$$a + b + c - 1 - (ab + ac - a) = a + b - ab + a + c - ac - 1$$

$$a + b + c - 1 - ab - ac + a = a + b - ab + a + c - ac - 1$$

$$2a + b + c - 1 - ab - ac = 2a + b - ab + c - ac - 1$$

4. $(\mathbb{Z}, \circ)$ is Comm. ring.

$$a \circ b = a + b - ab$$

$$= b + a - ab$$

$$= b \circ a$$

5. $(\mathbb{Z}, \circ)$ is identity.

$$a \circ e = a$$

$$a + e - ae = a$$

$$e - ae = a - a$$

$$e(1 - a) = 0$$

$$e = 0$$

$(\mathbb{Z}, *, \circ)$ is Comm. ring with identity

Zero divisor

قواسم صفرية

Def:- let $(R, +, \cdot)$ be a ring. An element $a \in R$ is called Zero divisor if $a \neq 0$ and there exists $b \neq 0$ s.t. $a \cdot b = 0$.

* يوجد عددان في (R) حيث $a \neq 0, b \neq 0$ $a \cdot b = 0$

Example:- $(\mathbb{Z}_6, +, \cdot)$ Ring. find Zero divisor?

Sol: $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$

$$a \neq 0 \rightarrow 2 \neq 0$$

$$b \neq 0 \rightarrow 3 \neq 0$$

$$\therefore 2 \cdot 3 = 6 - 6 = 0$$

$$\text{Zero divisor} = \{2, 3\}$$

$$a \neq 0 \rightarrow 3 \neq 0$$

$$b \neq 0 \rightarrow 4 \neq 0$$

$$\therefore 3 \cdot 4 = 12 - 6 - 6 = 0$$

$$\text{Zero divisor} = \{3, 4\}$$

* في $\mathbb{Z}_n$ إذا كان n

عدداً زوجياً فأن $\mathbb{Z}_n$

لا يمتلك قواسم

صفرية

Prime: هو العدد الذي

Example:- $(\mathbb{Z}_5, +, \cdot)$ Ring. find zero?

يفصل القسمة على نفسه

بأنه الواحد فقط مثال

$\{1, 3, 5, 7\}$

Sol: $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$

is not zero divisor

$$\nexists a \neq 0, b \neq 0, a \cdot b = 0$$

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$(\mathbb{Z}, +, \cdot)$
 $(R, +, \cdot)$ is not zero divisor
 $(Q, +, \cdot)$

SHAFAGH

صباح

أو
خطأ

Date: _____

Q. $(\mathbb{Z}_8, +, \cdot)$ Find Zero divisor?

Sol: $\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$

$a = 2 \neq 0$ st: $a \cdot b = 0$

$b = 4 \neq 0$

$2 \cdot_8 4 = 8 - 8 = 0$

$a = 6 \neq 0$

$b = 4 \neq 0$

$6 \cdot_8 4 = 24 - 8 = 16 - 8 = 8 - 8 = 0$

is Zero divisor = $\{2, 4, 6\}$

Example: let $(R \times R, +, \cdot)$ be triple ordered

$(a, b) + (c, d) = (a+c, b+d)$

$(a, b) \cdot (c, d) = (ac, bd)$

Show that $(R \times R, +, \cdot)$ is Comm. ring with identity and find Zero divisor?

Sol: 1. $(R \times R, +, \cdot)$ is Comm. group.

a. $(+)$ is closed from def.

b. $\forall (a, b), (c, d), (e, f) \in R \times R$

$(a, b) + [(c, d) + (e, f)] = [(a, b) + (c, d)] + (e, f)$

$(a, b) + (c+e, d+f) = (a+c, b+d) + (e, f)$

$(a+c+e, b+d+f) = (a+c+e, b+d+f)$

c. $e = (e_1, e_2) = (0, 0)$

$(a, b) + (0, 0) = (a, b)$

d. $(a, b)^{-1} = (-a, -b)$

st $(a, b) + (a, b)^{-1} = (0, 0)$

$(a, b) + (-a, -b) = (0, 0)$

e. $(a, b) + (c, d) = (a+c, b+d) = (c, d) + (a, b)$

2. $(R \times R, \cdot)$ is Semi group.

a. $(\cdot)$ is closed from Def

b. $\forall (a, b), (c, d), (e, f)$

$$[(a, b)(c, d)](e, f) = (a, b)[(c, d)(e, f)]$$

$$(ace, bdf) = (ace, bdf)$$

3. $(\cdot)$ dist $(+)$

$$(a, b) \cdot [(c, d) + (e, f)] = [(a, b)(c, d)] + [(a, b)(e, f)]$$

$$(a, b) \cdot (c+e, d+f) = (ac, bd) + (ae, bf)$$

$$(ac+ae, bd+bf) = (ac+ae, bd+bf)$$

4. $(R \times R, \cdot)$ is commutative

$$(a, b)(c, d) = (c, d)(a, b)$$

$$(ac, bd) = (ca, db)$$

5. $(R \times R, \cdot)$ is identity

$$(a, b)(1, 1) = (a, b)$$

$$e = (1, 1) \in R \times R$$

$(R \times R, +, \cdot)$ is Comm. ring

is Zero divisor =

$$(a, b)(c, d) = (0, 0)$$

$$(0, b)(c, 0) = (0, 0)$$

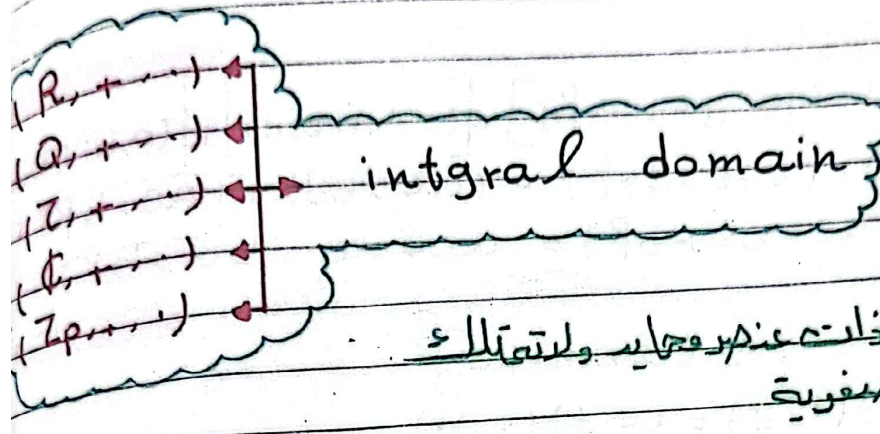
$$(a, 0)(0, d) = (0, 0)$$

$$\therefore \text{Zero divisor} = \{(0, b), (c, 0), (a, 0), (0, d)\}$$

Def: An integral domain ... شروط الطقة التامة ...

1- طقة ابدال ذات عنصر محايد comm. ring with identity and

2- لا تمتلك قواسم صفرية without zero divisor.



* طقة ابدال ذات عنصر محايد ولا تمتلك قواسم صفرية

example: $(\mathbb{Z}_6, +, \cdot)$ integral domain?

Sol: 1. $(\mathbb{Z}_6, +, \cdot)$ is comm. ring with identity

2. $2 \neq 0, 3 \neq 0, 4 \neq 0$

$2 \cdot 3 = 6 - 6 = 0$

$3 \cdot 4 = 12 - 6 = 6 - 6 = 0$

zero divisor $\{2, 3, 4\}$

$\therefore$ is not integral domain.

Q/ give example ring have not integral?

Sol: $(\mathbb{Z}_2, +, \cdot)$ because have Comm. ring without identity

$(\mathbb{R} \times \mathbb{R}, +, \cdot)$ is not integral domain because have zero divisor

Theorem 1 A ring $(R, +, \cdot)$ is without zero divisor
iff the cancellation law hold for
multiplication.

المجموعة R لا تحتوي على قواسم صفرية
إذا وفقط إذا قانون الحذف تحقق
لعلمية الضرب.

Proof: we assume that

R without zero divisor and let $a, b, c \in R$

$$\begin{aligned} \text{s.t } a \neq 0, ab = ac &\Rightarrow ab - ac = 0 \\ &\Rightarrow a(b - c) = 0 \\ &\Rightarrow a \neq 0, b - c = 0 \\ &\Rightarrow b = c \end{aligned}$$

Conversely: Suppose that the Cancellation
law hold and $a, b \in R$

$$\text{s.t } a \cdot b = 0, a \neq 0 \Rightarrow ab = a \cdot 0$$

$$b = 0$$

$$b \neq 0 \Rightarrow ab = ob$$

$$\Rightarrow a = 0$$

$\therefore R$ without zero divisor

Sub Ring

حلقة جزئية

Def:- let $(R, +, \cdot)$ be ring and $\emptyset \neq S \subseteq R$
then $(S, +, \cdot)$ is subring iff
s.t $a - b \in S$
 $a \cdot b \in S \quad \forall a, b \in S$

شروط Subring

$$a - b \in S, \quad \forall a, b \in S$$

$$a \cdot b \in S, \quad \forall a, b \in S$$

النوع الأول

* يأتي الموضوع على

نوعين من الأسئلة

1. إثباته

2. استخراج

Example:- $(\mathbb{Z}_e, +, \cdot)$ is Sub Ring of ring $(\mathbb{Z}, +, \cdot)$?

Sol, 1, let $a = 2n, b = 2m$

$$\forall a, b \in \mathbb{Z}_e$$

$$a - b \rightarrow 2n - 2m \rightarrow 2(n - m) \rightarrow 2r \in \mathbb{Z}_e$$

$$2, a \cdot b \rightarrow 2n \cdot 2m \rightarrow 4(nm) \rightarrow 4s \in \mathbb{Z}_e$$

Example:- $(\mathbb{Z}_0, +, \cdot)$ is not Subring of ring $(\mathbb{Z}, +, \cdot)$?

Sol, let $a = 7, b = 3 \quad \forall a, b \in \mathbb{Z}_0$

$$a - b \rightarrow 7 - 3 \rightarrow 4 \notin \mathbb{Z}_0$$

Example:- $(\mathbb{Z}, +, \cdot)$ is Subring of ring $(\mathbb{Q}, +, \cdot)$?

Sol, 1, let $a = 3, b = -4 \quad \forall a, b \in \mathbb{Z}$

$$a - b \rightarrow 3 - (-4) \rightarrow 3 + 4 \rightarrow 7 \in \mathbb{Z}$$

$$2, a \cdot b \rightarrow 3 \cdot (-4) \rightarrow -12 \in \mathbb{Z}$$

HAFAH

Trivial Subring and proper subring

النوع الثاني

Example:- Find all subring of ring $(\mathbb{Z}_{12}, +_{12}, \cdot_{12})$?

$$12 = 2 \cdot 6, 12 = 3 \cdot 4$$

Sol,

$$S_1 = \{ \mathbb{Z}_{12}, +_{12}, \cdot_{12} \}$$

$$S_2 = \{ 0, +_{12}, \cdot_{12} \}$$

$$S_3 = \{ 0, 2, 4, 6, 8, +_{12}, \cdot_{12} \}$$

$$S_4 = \{ 0, 3, 6, 9, +_{12}, \cdot_{12} \}$$

$$S_5 = \{ 0, 4, 8, +_{12}, \cdot_{12} \}$$

$$S_6 = \{ 0, 6, +_{12}, \cdot_{12} \}$$

trivial تافهة

proper

فولية

إذا كانت n عدد أولي فإن المجموعات تقتصر على التافهة فقط.
إذا كانت n عدد مركب فإن المجموعات تكون فولية وتافهة.

Example:- Find all subring of ring $(\mathbb{Z}_6, +_6, \cdot_6)$?

$$6 = 2 \cdot 3$$

$$S_1 = \{ \mathbb{Z}_6, +_6, \cdot_6 \}$$

$$S_2 = \{ 0, +_6, \cdot_6 \}$$

$$S_3 = \{ 0, 2, 4, +_6, \cdot_6 \}$$

$$S_4 = \{ 0, 3, +_6, \cdot_6 \}$$

$$a - b \in \mathbb{Z}_6$$

$$4 - 3 = 1 \in \mathbb{Z}_6$$

Example:- Find all subring of ring $(\mathbb{Z}_{11}, +_{11}, \cdot_{11})$?

$$S_1 = \{ \mathbb{Z}_{11}, +_{11}, \cdot_{11} \}$$

$$S_2 = \{ 0, +_{11}, \cdot_{11} \}$$

Month: _____ Date: _____

Example 1 - Find all Subring in the following ring $(\mathbb{Z}_{33}, +_{33}, \cdot_{33})$?

Sol. $33 = 3 \times 11$

$$S_1 = \{ \mathbb{Z}_{33}, +_{33}, \cdot_{33} \}$$

$$S_2 = \{ 0, +_{33}, \cdot_{33} \}$$

$$S_3 = \{ 0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30 \}$$

$$S_4 = \{ 0, 11, 22, +_{33}, \cdot_{33} \}$$

Example 2 - Let $(R, +, \cdot)$ is ring and $S = [0] \times R$ is $([0] \times R, +, \cdot)$ Subring?

Sol. let $(0, a), (0, b) \in [0] \times R$

- 1, $(0, a) - (0, b) = (0, a - b) \in [0] \times R$
- 2, $(0, a)(0, b) = (0, a \cdot b) \in [0] \times R$

$\therefore S$ subring of $R \times R$

$$(R \times R, +, \cdot) \text{ identity} = (1, 1) = e_1$$

$$(0, 1) = e_2$$

Q / Zero divisor for $[0] \times R$?

Sol. No Zero divisor

$$\rightarrow (0, a)(0, b) = (0, ab) \neq (0, 0)$$

Q / give exampl subring not have Zero divisor but the ring have Zero divisor.

Sol. الجواب المثال
اخره كامل

اعطيه مثال على حلقة جزئية
لا تحتوي على قواسم صفرية
كن الحلقة تحتوي قواسم
صفرية.

Characteristic of Ring

.. مميز الحلقة ..

Def: let R be a ring if there exists a positive integer (n) s.t. $n \cdot a = 0$ $\forall a \in R$ then the least positive integer with this property is called the char of the ring

هو المميز عن جميع موجبات حيث اذا جمع مع نفسه n مراته سوف يساوي 0 $\forall a \in R$ $n \cdot a = 0$ $\forall a \in R$

ايه حلقة لا تستطيع الحصول على

المميز لها فانه المميز هو مميز الحلقة

Example: $(\mathbb{R}, +, \cdot)$

$(\mathbb{Q}, +, \cdot)$

$(\mathbb{Z}, +, \cdot)$

$(\mathbb{Z}_2, +, \cdot)$

$(\mathbb{Z}_3, +, \cdot)$

Char = 0

Example: $(\mathbb{Z}_2, +, \cdot)$ is Ring find Char?

في $\mathbb{Z}_2$ المميز الحلقة يساوي n دائماً

Sol, $\mathbb{Z}_2 = \{0, 1\}$

$$a=0, a=1, n=2$$

$$n \cdot a = 0$$

$$2 \cdot 0 = 0$$

$$2 \cdot 1 = 2 = 0$$

$$\therefore \text{Char} = \{2, 0\}$$

$$Z_4 = \{0, 1, 2, 3\}$$

$$Z_4 = \{0, 1, 2, 3\}$$

$$p \cdot a = 0$$

4.0-0

~~$4.1 = 4 - 4 = 0$~~

$$4.2 = 8 - 4 = 4 - 4 = 0$$

$$4 \cdot 3 = 12 - 4 = 8 - 4 = 4 - 4 = 0$$

$$\therefore \text{Char} = \{4, 0\}$$

Example: $(Z_5, +, \cdot)$ is ID? find char?

Sol. $Z_5 = \{0, 1, 2, 3, 4\}$

~~$n \cdot a = 0$~~

$$5.0 = 0$$

$$5.1 - 5 - 5 = 0$$

$$5.2 = 10 - 5, 5 - 5 = 0$$

$$5 \cdot 3 = 15 - 5 = 10 - 5 = 5 - 5 = 0$$

5. 4 = 20 - 5 - 5 - 5 - 5 - 5 = 0

Theorem example Char = $\{5, 0\}$

Example: $(Z_6, +_6, *_6)$ is ID find char?

Sol, $Z_6 = \{0, 1, 2, 3, 4, 5\}$

$$h.a = 0$$

6.020

$$6.1 = 6 - 6 = 0$$

$$6.9 = 12 - 6 = 6 - 6 = 0$$

$$6-3=18-6=12-6=6-6=0$$

SHAFAGH

$$\text{char} = \{0, 6\}$$

- Nil potent element -

- عنصر معدوم القوى -

def: let $(R, +, \cdot)$ be a ring and let $a \in R$
then a is called Nil potent
element if there exist $n \in \mathbb{Z}^+$
s.t. $a^n = 0$, $a \in R$

ليكن R حلقة $a \in R$ فيسمى a عنصراً معدوم القوى
إذا وجد عنصر موجب في $\mathbb{Z}$ حيث $a^n = 0$

Example 1. $(\mathbb{Z}_8, +, \cdot)$ Find Nil?

Sol, $\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$

$$a \in \mathbb{Z}_8$$

$$a^n = 0$$

$$2^3 = 8 - 8 = 0$$

$$4^2 = 16 - 8 - 8 = 0$$

$$6^3 = 216 - 8 - 8 - \dots = 0$$

$$\therefore \text{Nil} = \{0, 2, 4, 6\}$$

- Center of Ring - - مركز الحلقة -

Def 1: let $(R, +, \cdot)$ be a ring then the center of R is defined as

$$\text{Cent}(R) = \{c \in R : cx = xc, \forall x \in R\}$$

كل عناصر الحلقة ابدالية لعملية الضرب

يكن حلقة ابدالية مركزها يساوي الحلقة نفسها

Example:- Find Cent

a, $(R, +, \cdot) \rightarrow \text{Cent}(R) = R$

b, $(Q, +, \cdot) \rightarrow \text{Cent}(Q) = Q$

c, $(Z, +, \cdot) \rightarrow \text{Cent}(Z) = Z$

d, $(Z_n, +, \cdot) \rightarrow \text{Cent}(Z_n) = Z_n$

Example:- $(Z_{10}, +, \cdot)$ is ring. Find Cent?

Sol, $Z_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$\text{Cent}(Z_{10}) = Z_{10}$

Example:- $(Z_6, +, \cdot)$ is ring. Find Cent?

Sol, $Z_6 = \{0, 1, 2, 3, 4, 5\}$

$\text{Cent}(Z_6) = Z_6$

Example:- $(M_2(Z), +, \cdot)$ is ring. Find Cent?

Sol, $\text{Cent}(M_2(Z)) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Idem potent element

عنصر متساوي القوى

Ex: let $(R, +, \cdot)$ be a ring and let $a \in R$ then a is said to be idem potent element if $a^2 = a$

يكونه الحلقة ذاته عنصراً متساوياً للقوى
وكل عنصر فيها قريباً يساوي نفسه

Example: $(\mathbb{Z}_2, +, \cdot)$ Find Idem potent element? $\forall a \in \mathbb{Z}_2, a^2 = a$

Sol, $\mathbb{Z}_2 = \{0, 1\}$

$$a^2 = a \rightarrow (0)^2 = 0$$

$$\rightarrow (1)^2 = 1$$

$$\therefore \mathbb{Z}_2 \text{ is Idem} = \{0, 1\}$$

Example: $(\mathbb{Z}_4, +, \cdot)$ Find Idem potent?

Sol, $\mathbb{Z}_4 = \{0, 1, 2, 3\}$

$$a^2 = a \rightarrow (0)^2 = 0$$

$$(1)^2 = 1$$

$$(2)^2 = 4$$

$$(3)^2 = 6$$

$$\therefore \text{Idem potent} = \{0, 1\}$$

Example 1 - prove that $(\text{cent}(R, +, \cdot))$ is sub ring of (R) ?

Sol. 1. $\text{Cent}(R) \neq \emptyset$ [since $e \in \text{cent}(R)$]

2. let $a, b \in \text{cent}(R)$, $x \in R$

$$\text{Cent}(R) \rightarrow \begin{aligned} ax &= xa \\ bx &= xb \end{aligned}$$

$$\begin{aligned} (a-b)x &= ax - bx \\ &= xa - xb \\ &= x(a-b) \\ \therefore a-b &\in \text{cent}(R) \end{aligned}$$

$$\begin{aligned} (a \cdot b)x &= x(a \cdot b) \\ (a \cdot b)x &= a(bx) \\ &= a(xb) \\ &= (ax)b \\ &= (xa)b \\ &= x(ab) \end{aligned}$$

$$\therefore (ab) \in \text{cent}(R)$$

تعتبر فريدة

Ideal المثالي

Def: let $(R, +, \cdot)$ be a ring and let $\emptyset \neq I \subseteq R$ then $(I, +, \cdot)$ is called right ideal if

- شروط
right
1. $a \cdot b \in I, \forall a, b \in I$
 2. $a \cdot r \in I, r \in R, a \in I$

and called left ideal if

- شروط
left
1. $a \cdot b \in I, \forall a, b \in I$
 2. $r \cdot a \in I, r \in R, a \in I$
- باني الشرط الثاني
شكلا left
أو right على
شكلا فئات

and called ideal if $(I, +, \cdot)$ is left and right.

* كل مثالي هو حلق جزئية والعكس ليس بالضرورة.
* اتحاد مثاليين ليس بالضرورة ان يكون مثالي.

Example: Show that $(\mathbb{Z}, +, \cdot)$ is an ideal of ring $(\mathbb{Q}, +, \cdot)$?

let $a=1, b=2, \forall a, b \in \mathbb{Z}$

- Sol,
1. $a \cdot b \in \mathbb{Z}$
 $1 \cdot 2 = 2 \in \mathbb{Z}$
 2. let $a \in \mathbb{Z}, r \in \mathbb{Q}$
 $r \cdot a = \frac{1}{3} \cdot 2 = \frac{2}{3} \notin \mathbb{Z}$

$\therefore \mathbb{Z}$ is not ideal of $\mathbb{Q}$

Example: - Show that $(Z_6, +, \cdot)$ is an ideal of ring $(Z, +, \cdot)$?

Sol. let $a = 2n, b = 2m, \forall a, b \in Z_6$

1, $a - b \in Z_6$

$$2n - 2m = 2(n - m) = 2r \in Z_6$$

2, let $a = 2n, \forall a \in Z_6$

$$r = 3, \forall r \in Z$$

left: $ra = 3 \cdot 2n = 6n \in Z_6$

right: $ar = 2n \cdot 3 = 6n \in Z_6$

$\therefore Z_6$ is Ideal of Z

Trivial ideal and proper ideal
فجوة متافئة

Example 1: Find all ideal of the ring $(Z_5, +, \cdot)$

Sol. $Z_5 = \{0, 1, 2, 3, 4\}$

$$I_1 = \{0, +_5, \cdot_5\}$$

$$I_2 = \{0, +_5, \cdot_5\}$$

Example: - $I = \{1, 5, 7, 11\}$ is Ideal?

Sol. let $a = 5, b = 1, \forall a, b \in I$

$$\therefore a - b \in I$$

$5 - 1 = 4 \notin I \therefore I$ is not Ideal

* الحلقة التي تحتويه

على مثاليات تافئة

فقط تكون حلقة بسيطة

* في الحلقة Z_n اذا كان

n عددا اوليا فانه الحلقة

تحتوي على مثاليات

تافئة فقط.

Example:- Find all ideal of the ring

$(\mathbb{Z}_{12}, +_{12}, \cdot_{12})$?

Sol,

$$I_1 = \{ \mathbb{Z}_{12}, +_{12}, \cdot_{12} \}$$

2x6
3x4

$$I_2 = \{ (0), +_{12}, \cdot_{12} \}$$

$$I_3 = \{ (0, 2, 4, 6, 8, 10), +_{12}, \cdot_{12} \}$$

$$I_4 = \{ (0, 3, 6, 9), +_{12}, \cdot_{12} \}$$

$$I_5 = \{ (0, 4, 8), +_{12}, \cdot_{12} \}$$

$$I_6 = \{ (0, 6), +_{12}, \cdot_{12} \}$$

$\mathbb{Z}_{12}$ is Ideal

Example:- Find all ideal of the ring

$(\mathbb{Z}_{24}, +_{24}, \cdot_{24})$?

Sol,

$$I_1 = \{ \mathbb{Z}_{24}, +_{24}, \cdot_{24} \}$$

$$I_2 = \{ (0), +_{24}, \cdot_{24} \}$$

$$I_3 = \{ (0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22), +_{24}, \cdot_{24} \}$$

$$I_4 = \{ (0, 3, 6, 9, 12, 15, 18, 21), +_{24}, \cdot_{24} \}$$

$$I_5 = \{ (0, 4, 8, 12, 16, 20), +_{24}, \cdot_{24} \}$$

$$I_6 = \{ (0, 6, 12, 18), +_{24}, \cdot_{24} \}$$

$$I_7 = \{ (0, 8, 16), +_{24}, \cdot_{24} \}$$

$$I_8 = \{ (0, 12), +_{24}, \cdot_{24} \}$$

$\mathbb{Z}_{24}$ is ideal

Example: $(\mathbb{R}, +, \cdot)$ is ring, $I = (\mathbb{Z}, +, \cdot)$ is Ideal?

Sol, 1. $a - b \in \mathbb{Z}, \forall a, b \in \mathbb{Z}$

2. $r \in \mathbb{R}$ real

$a \in \mathbb{Z}$ intg

let $a = 7 \quad \forall a \in \mathbb{Z}$

$r = \frac{1}{3} \quad \forall r \in \mathbb{R}$

$a \cdot r \in \mathbb{Z}$

$$7 \cdot \frac{1}{3} = \frac{7}{3} \notin \mathbb{Z}$$

$\therefore (\mathbb{Z}, +, \cdot)$ is not Ideal But $(\mathbb{Z}, +, \cdot)$ Subring.

Q/give example not Ideal But is Subring?

Sol, let $(\mathbb{R}, +, \cdot)$ is ring

$(\mathbb{Z}, +, \cdot)$ is Subring not Ideal

Q/give me example:-

اكتب مثال كل مايلي

1. Sub ring not ideal

Sol, $(\mathbb{R}, +, \cdot)$ is ring

2.

$(\mathbb{Z}, +, \cdot)$ is Sub ring

3.

not Ideal

4.

استل عن فوضف
افضل

Date: _____
 Theorem: - let I be a proper ideal of ring $(R, +, \cdot)$ with identity then no element of I has a multiplicative inverse.

كل مثالي فعلي لربطتين على معكوس فربطتين
 لا يملك عنده من العناصر.

of, let $a \neq 0 \in I$
 suppose that $a^{-1} \in I$
 $aa^{-1} = 1 \in I$
 $\forall r \in R$
 $1 \cdot r = r$
 $r \in I \rightarrow R \subset I$
 by Def $I \subseteq R \rightarrow R = I$

Example for Theorem
 $R = \mathbb{Z}_5, I = \{1, 5, 7, 11\}$

Sol, $5 \cdot 5 = 25 = 24 + 1$
 $5 \cdot 5^{-1} = 5 \cdot \frac{1}{5} = 1$
 $\therefore$ not Ideal.

$$\begin{aligned}
 & (2, 1, +) \\
 & a + (-a) = 0
 \end{aligned}$$

Theorem 1. let I_1 and I_2 are ideal of Ring $(R, +, \cdot)$, Then $I_1 \cap I_2$ is ideal of R .

proof 1. 1. $I_1 \cap I_2 \neq \emptyset \rightarrow$ Since $\{e\} \in I_1 \cap I_2$

2. $a, b \in I_1 \cap I_2$

$\Rightarrow a, b \in I_1$ and $a, b \in I_2$

$\rightarrow I_1$ ideal $\rightarrow a - b \in I_1$

I_2 ideal $\rightarrow a - b \in I_2$

$(a - b) \in I_1 \cap I_2$

3. let $a \in I_1 \cap I_2$

$a \in I_1$ and $a \in I_2$, $r \in R$

$\rightarrow I_1$ ideal $\rightarrow ar \in I_1$

I_2 ideal $\rightarrow ar \in I_2$

$\Rightarrow ar \in I_1 \cap I_2$

$\Rightarrow I_1 \cap I_2$ ideal

$\rightarrow ar \in I_1 \cap I_2$

$ra \in I_1$

$ra \in I_2$

$ra \in I_1 \cap I_2$

$\Rightarrow I_1 \cap I_2$ ideal

Example 1: in Z_{12} — example of Theorem.

Sol, $I_1 = \{0, 2, 4, 6, 8, 10\}$

$I_3 \cap I_4 = \{0\}$

$I_2 = \{0, 6\}$

$I_2 \cap I_3 = \{0, 6\}$

$I_3 = \{0, 3, 6, 9\}$

$I_1 \cap I_4 = \{0, 4, 8\}$

$I_4 = \{0, 4, 8\}$

Example 1 in Z_{20} — Find $Z_{20} \cap I_3 \cap I_4$?

4x5
2x10

Example: let $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, a, b, c, d \in R \right\}$

$(R, +, \cdot)$ is a ring with identity, and

$I_1 = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, a, b \in R \right\}$, then show

that I_1 is left or right ideal?

sol. 1. $\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix} \in I_1, \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix} \in I_1, \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 & 0 \end{bmatrix} \in I_1$

2. $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in R, \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix} \in I_1$

right
or $\rightarrow \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} a_2 a_1 + b_2 c_1 & a_2 b_1 + b_2 d_1 \\ 0 & 0 \end{bmatrix} \in I_1$

I_1 right ideal, $a \in I, r \in R, ar \in I$

left
or $\rightarrow \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 a_2 & a_1 b_2 \\ c_1 a_2 & c_1 b_2 \end{bmatrix} \notin I$

I_1 is not left ideal.
 $r a$ is not left ideal.

Q/give me example for ideal not left but is right with Solv?

اعطيه مثال لا يحقق من left
لكن يحقق من right مع
الدلائل.

* اذا جاء في السؤال كلمة (Solv)
فعلما وشبهه مع المثال

Def: let R be a ring and $a \in R$
 $(a) = aR = [ar : r \in R]$ is right principal ideal
 $(a) = (Ra) = [ra : r \in R]$ is left principal ideal

هو المثال الذي يتولد بعنصر مفرد (a)

Example: the principal ideal of the ring $(\mathbb{Z}, +, \cdot)$?

Sol: $(2) = 2\mathbb{Z} = \{0, \pm 2, \pm 4, \pm 6, \dots\}$
 $(3) = 3\mathbb{Z} = \{0, \pm 3, \pm 6, \pm 9, \dots\}$
 $(n) = n\mathbb{Z} = \{0, \pm n, \pm n+n, \dots\}$

Example 1: the principal ideal of the ring $(\mathbb{Z}_{12}, +, \cdot)$

$$12 = 2 \cdot 6$$

$$12 = 3 \cdot 4$$

Sol: $I_1 = (1) = \{0, 1, 2, \dots, 11\}$ ← trivial
 $I_2 = (0) = \{0\}$ ←
 $I_3 = (2) = \{0, 2, 4, 6, 8, 10\}$ ←
 $I_4 = (3) = \{0, 3, 6, 9\}$ ←
 $I_6 = (4) = \{0, 4, 8\}$ ←
 $I_6 = (6) = \{0, 6\}$ ← proper

principal ideal Ring

حلقة المثالي الرئيسية

Def: - An ring R is said to be principal ideal Ring, if R is a comm. ring with identity in which every ideal of R is principal.

1. حلقة ابدالية .

2. تحتويه هوية .

3. كل مثالي اساسيه .

Example: the ring $(\mathbb{Z}, +, \cdot)$ principal Ideal Ring?

Sol, (1) $= \mathbb{Z}$

(2) $= \{0, \pm 2, \dots\}$

(3) $= \{0, \pm 3, \dots\}$

والانتهاء من المثاليات الرئيسية

في حلقة $(\mathbb{Z}, +, \cdot)$

* لان كل مثالي فيها

استطيع ان اكتبه

بصورة عنصر واحد .

Q/ give example principal ideal Ring?

Sol, $(\mathbb{Z}, +, \cdot)$

$(\mathbb{Z}_2, +, \cdot)$

$(\mathbb{Z}_{24}, +, \cdot)$



principal ideal Ring

Quotient Ring

حلقة القسمة

Def: let R be a ring and I is an ideal of R then $a+I$
 $a+I = [a+i : i \in I]$ is coset of I in the ring R and

شروط حلقة القسمة

1. $(a+I) \oplus (b+I) = (a+b)+I$
2. $(a+I) \otimes (b+I) = (ab)+I$

Theorem 1- if I is an ideal of ring $(R, +, \cdot)$ then $(R/I, \oplus, \otimes)$ is a ring which is called (quotient ring of R by I) s.t $R/I = [a+I, a \in R]$.

Proof 1. $(R/I, \oplus)$ is Comm. group.

1. let $(a+I), (b+I) \in R/I$
 $\therefore (a+I) \oplus (b+I) \in R/I \therefore \oplus$ is closed

$$\begin{aligned} [(a+I) \oplus (b+I)] \oplus (c+I) &= (a+I) \oplus [(b+I) \oplus (c+I)] \\ (a+b)+I \oplus (c+I) & \\ (a+b+c)+I & \\ a+(b+c)+I & \\ (a+I) \oplus (b+c)+I & \\ (a+I) \oplus [(b+I) \oplus (c+I)] & \end{aligned}$$

$$\begin{aligned} \exists 0 \in R \text{ s.t } 0+I &\in R/I \\ (a+I) \oplus (0+I) &= (a+0)+I \\ &= a+I \in R/I \end{aligned}$$

$$4- \forall a+I, \exists -a+I \in R/I$$

$$(a+I) \oplus (-a+I) = (0+I)$$

$$(a-a)+I = (0+I)$$

$$(0+I) = (0+I)$$

$$5- (a+I) \oplus (b+I) = (a+b)+I$$

$$= (b+a)+I$$

$$= (b+I) \oplus (a+I)$$

2. $(R/I, \otimes)$ is semi group

1. $(a+I) \otimes (b+I) \in R/I$ $\otimes$ is closed

$$2- (a+I) \otimes [(b+I) \otimes (c+I)] = [(a+I) \otimes (b+I)] \otimes (c+I)$$

$$(a+I) \otimes (bc+I)$$

$$(abc+I)$$

$$(ab+I) \otimes (c+I)$$

$$[(a+I) \otimes (b+I)] \otimes (c+I)$$

H.W

3. $\otimes$ dist $\oplus$

$$(a+I) \otimes [(b+I) \oplus (c+I)] = [(a+I) \otimes (b+I)] \oplus [(a+I) \otimes (c+I)]$$

$$(a+I) \otimes [(b+c)+I]$$

$$= a(b+c) + I$$

$$= (ab+ac) + I$$

$$= (ab+I) \oplus (ac+I)$$

نبي بالطرف اليسر

ونزل بالطرف اليمين

$$= [(a+I) \otimes (b+I)] \oplus [(a+I) \otimes (c+I)]$$

Example :- let $R = (\mathbb{Z}, +, \cdot)$ Find $\mathbb{Z}/(2), \mathbb{Z}/(3)$?
 let $I = (2) = [0, \pm 2, \pm 4, \dots]$

Sol $\mathbb{Z}/(2) = [(0+(2)), (1+(2))]$

حسب 1

$\mathbb{Z}/(3) = [0, \pm 3, \pm 6, \pm 9, \dots]$

تكتب عدد جبروتات

Sol $\mathbb{Z}/(3) = [(0+(3)), (1+(3)), (2+(3))]$

Example :- let $(\mathbb{Z}_{12}, +, \cdot)$, $I = (2) = [0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10]$
 Find $\mathbb{Z}_{12}/(2), \mathbb{Z}_{12}/(6)$?

Sol $\mathbb{Z}_{12}/(2) = [(0+(2)), (1+(2))]$

$\mathbb{Z}_{12}/(6) = [(0+(6)), (1+(6)), (2+(6)), (3+(6)), (4+(6)), (5+(6))]$

Theorem: if $(R, +, \cdot)$ is ring with identity then $(R/I, \oplus, \otimes)$ is a ring with identity.
 اثبت ان R/I هو حلقة ذات عنصر هوية.

proof. Since R is ring

$$\exists 1 \in R, \text{ s.t. } a \cdot 1 = 1 \cdot a = a \in R$$

$$\therefore a + I \rightarrow (a \cdot 1) + I = (a + I) \otimes (1 + I) \xrightarrow{R/I} (a + I) \in R/I$$

$$\therefore \exists (1 + I) \in R/I$$

$$\text{S.t. } (a + I) \otimes (1 + I) = (a + I)$$

$\therefore R/I$ is ring with identity.

Example: let $R = (\mathbb{Z}_6, +, \cdot)$ is a ring without identity find $\mathbb{Z}_6/(4)$?

$$\text{Sol } \mathbb{Z}_6 = \{0, \pm 2, \pm 4, \pm 6, \pm 8, \dots\}$$

$$I = (4) = \{0, \pm 4, \pm 8, \dots\}$$

$$\mathbb{Z}_6/(4) = \{0 + (4), 2 + (4), \dots\}$$

$\therefore \mathbb{Z}_6/(4)$ quotient ring without identity
 $1 + I \notin \mathbb{Z}_6/(4)$

Q/ give me example quo. ring without identity
Sol $(\mathbb{Z}_6, +, \cdot)$

Def:- let I_1 and I_2 be Ideal of Ring (R)

$$I_1 + I_2 = [a+b, a \in I_1, b \in I_2]$$

اسم التعريف

Example:- let $R = (\mathbb{Z}_{12}, +, \dots)$

$$\text{let } I_1 = (3) = [0, 3, 6, 9]$$

$$I_2 = (6) = [0, 6] \quad \text{find } I_1 + I_2$$

$$I_3 = (4) = [0, 4, 8] \quad \text{find } I_1 + I_3$$

$$I_1 + I_2 = [0, 3, 6, 9] = I_1$$

$$I_1 + I_3 = [0, 3, 6, 9, 4, 7, 10, 1, 8, 11, 2, 5] = \mathbb{Z}_{12}$$

* نجمع كل عنصر من I_2 مع جميع عناصر I_1

* نجمع كل عنصر من I_3 مع جميع عناصر I_1

Theorem: let I_1 and I_2 are ideal of ring (R) . then $(I_1 + I_2, +, \cdot)$ is Ideal of $(R, +, \cdot)$

proof:

1. $I_1 + I_2 \neq \emptyset$ $[0 + 0 = 0 \in I_1 + I_2]$

2. let $(a_1 + b_1), (a_2 + b_2) \in I_1 + I_2$
 s.t $(a_1 + b_1) - (a_2 + b_2) = (a_1 - a_2) + (b_1 - b_2)$
 $\vdots$
 $\in I_1 + I_2$

3. $ar \in I$

let $(a + b) \in I_1 + I_2, r \in R$

left $r(a + b) = ra + rb \in I_1 + I_2$

right $(a + b)r = ar + br \in I_1 + I_2$

$\therefore (I_1 + I_2, +, \cdot)$ is ideal

Example: let $R = \mathbb{Z}, I_1 = 10\mathbb{Z}, I_2 = 100\mathbb{Z}$
 find $I_1 + I_2$?

Sol $I_1 + I_2 = 10\mathbb{Z} + 100\mathbb{Z} = 10(\mathbb{Z} + 10\mathbb{Z}) \in I_1 + I_2$

direct sum of I_1 and I_2
 جمع مباشر

Ex: let I_1 and I_2 be ideal of the Ring (R) , a ring (R) is said to be direct sum of I_1 and I_2 , if $I_1 + I_2 = R$ and $I_1 \cap I_2 = \{0\}$ and denoted by $(I_1 \oplus I_2 = R)$.

Example: let $R = (\mathbb{Z}_{12}, +, \cdot)$ is ring

$$I_1 = (3) = [0, 3, 6, 9]$$

$$I_2 = (4) = [0, 4, 8] \quad \text{Find } I_1 \cap I_2$$

is direct sum?

$$1, I_1 \cap I_2 = [0]$$

$$2, I_1 + I_2 = [0, 3, 6, 9, 4, 7, 10, 1, 8, 11, 2, 5]$$

$\therefore \mathbb{Z}_{12}$ is direct sum of I_1 and I_2

شروط الجمع المباشر

$$[0] = I_1 \cap I_2 \quad 1$$

$$R = I_1 + I_2 \quad 2$$

Def:- let I_1 and I_2 be ideal in the Ring $(R, +, \cdot)$ then $I_1 I_2$

$$I_1 \cdot I_2 = \sum_{i=1}^n a_i b_i, a_i \in I_1, b_i \in I_2$$

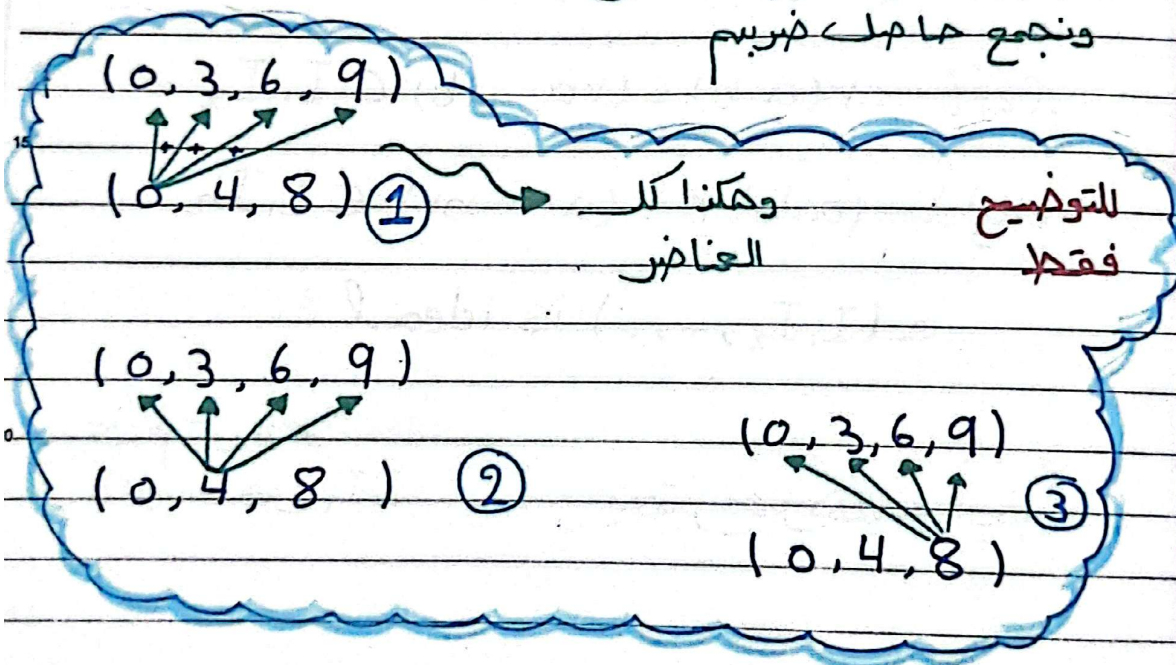
Example:- in $(\mathbb{Z}_{12}, +, \cdot)$ $I_1 = (3) = (0, 3, 6, 9)$

$$I_2 = (4) = (0, 4, 8)$$

Find $I_1 I_2$?

Sol: $I_1 I_2 = [0]$

بمنسوب به كل عنصر في I_1 بعناصر I_2
ونجمع حامل ضربهم



$$I_1 I_2 = [0.0 + 0.3 + 0.6 + 0.9 + 4.0 + 4.3 + 4.6 + 4.8 + 8.0 + 8.3 + 8.6 + 8.9]$$

$$= [0]$$

W :- prove that $(I_1, I_2, +, \cdot)$ is
Ideal of $(R, +, \cdot)$?

1. $I_1, I_2 \neq \emptyset$ [$0 \in I_1, I_2$]

2. let $(a, b_1), (a_2, b_2) \in I_1, I_2$

s.t. $(a, b_1) - (a_2, b_2) = (a_1 - a_2, b_1 - b_2) \in I_1, I_2$

3. $a, r \in I$

let $(a, b) \in I_1, I_2, r \in R$

left $r(a, b) = (ra, b) \in I_1, I_2$

right $(a, b)r = (a, br) \in I_1, I_2$

$\therefore (I_1, I_2, +, \cdot)$ is ideal

يعتبر فبرهنت ايضاً

نستخدم شروط ideal
على شكل العنصر I_1, I_2

the Ring homomorphism

Def: let $(R, +, \cdot)$ and $(\bar{R}, *, \circ)$ are rings and let $f: R \rightarrow \bar{R}$ be a function then f is said to be a ring homomorphism.

iff: - شروط

$$1 - f(a+b) = f(a) * f(b)$$

$$2 - f(a \cdot b) = f(a) \circ f(b)$$

Example: let $f: (\mathbb{Z}, +, \cdot) \rightarrow (\mathbb{Z}, +, \cdot)$ defined by $f(a) = 2a$, show that f is a ring homo or not?

نبدعي بالطرف اليسر للشروط ونحل الطرف اليمين

Sol, let $a, b \in \mathbb{Z}$

$$1 - f(a+b) = 2(a+b) = 2a + 2b = f(a) + f(b)$$

طرف يسار طرف يمين

$$2 - f(a \cdot b) = 2(a \cdot b) = 2a \cdot 2b = 4ab$$

نحقق بافتراض

$$f(a \cdot b) \neq f(a) \cdot f(b)$$

الشروط الاول

ونحقق بافتراض

f is not homo.

$$f(a \cdot b) = f(a) \cdot f(b)$$

$$2(a \cdot b) = 2a \cdot 2b$$

$$2a \cdot b \neq 4ab$$

Month _____ Date _____

Example:- let $(R, +, \cdot)$ be a ring with identity and iff

$f_a: R \rightarrow R$ defined as

$$f_a(x) = axa^{-1}, \forall x \in R, a \in R$$

Show that f_a is a ring homo?

sol 1. $x, y \in R$

$$\begin{aligned} f_a(x+y) &= a(x+y)a^{-1} \\ &= (ax+ay)a^{-1} \\ &= axa^{-1} + aya^{-1} \\ &= f_a(x) + f_a(y) \end{aligned}$$

حسب الشرط
نبدأ بالطرف اليمين ونحول إلى الطرف الأيسر
حرف أيسر
حرف أيسر

$$\begin{aligned} f_a(xy) &= a(xy)a^{-1} \\ &= (ax)ay^{-1} \\ &= axa^{-1}aya^{-1} \\ &= f_a(x) \cdot f_a(y) \end{aligned}$$

نفرض أن x و y بين a و a^{-1} وننتج $a^{-1}a$
حرف أيسر
حرف أيسر

$\therefore f_a$ is ring homo.

تعتبر هومومورف

The Kernel ring homo نواة حالة التشاكل الحلقية

Def:- let f be a ring homo. from the ring $(R, +, \cdot)$ in to the ring $(\bar{R}, +, \cdot)$ then the kernel of f is defined by

$$\text{Ker}(f) = \{a \in R : f(a) = \bar{0}\}$$

Example:-

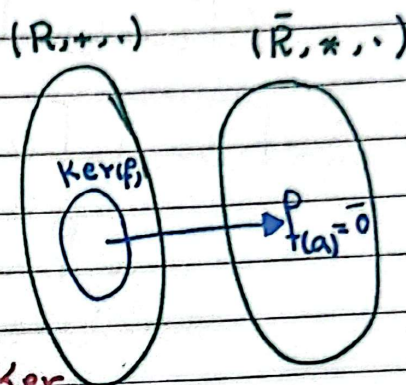
let $f: (\mathbb{Z}, +, \cdot) \rightarrow (\mathbb{Z}_2, +, \cdot)$

defined by

$$f(a) = \begin{cases} 0 & \text{if } a \in \mathbb{Z}_e \\ 1 & \text{if } a \in \mathbb{Z}_o \end{cases}$$

find Ker

prove that f is a ring homo



$$f(a) = 1 \quad a \in \mathbb{Z}_o$$

1. let $a, b \in \mathbb{Z}_e$

$$f(a+b) = 0+0 = 0 \in \mathbb{Z}_e$$

$$= f(a) + f(b)$$

$$f(a \cdot b) = 0 = 0 \cdot 0 \in \mathbb{Z}_e$$

$$= f(a) \cdot f(b)$$

2. let $a, b \in \mathbb{Z}_o$

$$f(a+b) = 1+1 = 2 = 0 \in \mathbb{Z}_e$$

$$= f(a) + f(b)$$

$$f(a \cdot b) = 1 \cdot 1 = 1 \in \mathbb{Z}_o$$

3. let $a \in \mathbb{Z}_e, b \in \mathbb{Z}_o$

$$f(a+b) = 0+1 = 1 \in \mathbb{Z}_o$$

$$= f(a) + f(b)$$

$$f(a \cdot b) = 0 \cdot 1 = 0 \in \mathbb{Z}_e$$

$$= f(a) \cdot f(b)$$

$\therefore f$ is ring homo

$$\text{Ker } f = \mathbb{Z}_e = 0$$

AFAGH

Theorem: let f be a ring homo from $(R, +, \cdot)$ into $(\bar{R}, \bar{+}, \bar{\cdot})$ then.

proof: 1. $f(0) = \bar{0}$, where $\bar{0}$ is the identity in $\bar{R}$

2. $f(-a) = -f(a)$, $\forall a \in R$

3. $f(R)$ is subring of $\bar{R}$

if f is R and $\bar{R}$ with identity

$f(R) = \bar{R}$ (onto) شاملة

4. $f(1) = \bar{1}$

5. $f(a^{-1}) = (f(a))^{-1}$, $\forall a^{-1} \in R$

15

* نتائج في الـ ring
على شكل شرح
وخطأ كخواص الـ ring

20

25

HAFAGH

Theorem: let f be a ring homo from $(R, +, \cdot)$ into $(\bar{R}, +, \cdot)$, then $\text{Ker}(f)$ is ideal of $(R, +, \cdot)$.

Proof: 1) $\text{Ker}(f) \neq \emptyset$
 $f(e) = \bar{e} \rightarrow$ حسب بروتة السابقة

* بما انه Ker موجود ونحاول ان نبرهن انه مثالي فنعلم ان العنصر 0 موجود في $\bar{R}$ وكما يجب المبرهنه السابقة $f(e) = \bar{e}$ للفرضه والجمع.

let $a, b \in \text{Ker}(f)$
 $\rightarrow f(a) = 0, f(b) = 0$
 $f(a-b) = f(a) - f(b)$
 $= 0 - 0$
 $= 0$

$\therefore f(a-b) = 0 \rightarrow a-b \in \text{Ker } f$

let $a \in \text{Ker}(f), r \in R$

$f(a) = 0$
 $f(ar) = f(a) \cdot f(r)$
 $= 0 \cdot f(r)$
 $= 0$

$\therefore f(ar) = 0 \rightarrow ar \in \text{Ker}(f)$

$\therefore \text{Ker}(f)$ is Ideal ■

Example: let $f: (\mathbb{Z}_4, +, \cdot) \rightarrow (\mathbb{Z}_2, +, \cdot)$
 def by $f(0) = f(2) = 0$
 $f(1) = f(3) = 1$
 Find $\mathbb{Z}_4 / \text{Ker}(f)$?

Sol :- to prove f is homo.

A. let $a = 0, b = 3$

1. $f(0+3) = f(3) = 1$

$f(0) + f(3) = 0 + 1 = 1$

$\therefore f(0+3) = f(0) + f(3)$

2. $f(0 \cdot 3) = f(0) = 0$

$f(0) \cdot f(3) = 0 \cdot 1 = 0$

$\therefore f(0 \cdot 3) = f(0) \cdot f(3)$

B. Now $a = 1, b = 2$

1. $f(1+2) = f(3) = 1$

$f(1) + f(2) = 1 + 0 = 1$

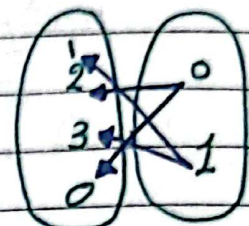
$\therefore f(1+2) = f(1) + f(2)$

2. $f(1 \cdot 2) = f(2) = 0$

$f(1) \cdot f(2) = 1 \cdot 0 = 0$

$\therefore f(1 \cdot 2) = f(1) \cdot f(2)$

$\therefore f$ is ring homo



دائماً $0 \in \text{Ker}(f)$
 لأن $f(0) = 0$

بما أنه من تعريف $\text{Ker}(f)$ و $0 \in \text{Ker}(f)$
 السؤال ذكر $0 = 2$ لهذا السبب ظهر

Now $\text{Ker}(f) = \{0, 2\}$

c) Now we must find

$\mathbb{Z}_4 / \text{Ker}(f) =$ طالع طلب السؤال

$0 + \text{Ker}(f) = \{0, 2\}$

$1 + \{0, 2\} = \{1, 3\}$

$2 + \{0, 2\} = \{2, 0\}$

$3 + \{0, 2\} = \{3, 1\}$

$\therefore \mathbb{Z}_4 / I = a + I, a \in \mathbb{Z}_4$

$\therefore \mathbb{Z}_4 / \text{Ker}(f) = [0 + \text{Ker}(f), 1 + \text{Ker}(f)]$

SHAFAGH

يتبع الحل

The natural mapping

Lemma:-

(nat_I) is a homomorphism function from the ring $(R, +, \cdot)$ on to the ring $(R/I, \oplus, \otimes)$ with $\text{Ker}(\text{nat}_I) = I$

of: let $\text{nat}_I: R \rightarrow R/I$ defined by

$$\text{nat}_I(a) = a + I$$

$$a \in R \quad a + I \in R/I$$

$$\forall a + I, b + I \in R/I, a, b \in R$$

we prove nat_I is homo.

$$\begin{aligned} 1. \text{nat}_I(a+b) &= (a+b) + I \\ &= (a+I) \oplus (b+I) \\ &= \text{nat}_I(a) \oplus \text{nat}_I(b) \end{aligned}$$

$$\begin{aligned} 2. \text{nat}_I(ab) &= (ab) + I \\ &= (a+I) \otimes (b+I) \\ &= \text{nat}_I(a) \otimes \text{nat}_I(b) \end{aligned}$$

$\therefore \text{nat}_I$ is homo fun.

now we find $\text{Ker}(\text{nat}_I)$ $\text{nat}_I: R \rightarrow R/I$

$$\begin{aligned} \text{Ker}(\text{nat}_I) &= \{a \in R : \text{nat}_I(a) = I\} \\ &= \{a \in R : a + I = I\} \\ &= \{a \in R : a \in I\} = I \end{aligned}$$

$$\begin{aligned} 0 + I &= I \\ 1 \cdot I &= I \end{aligned}$$

عنصر محايد لدالة القسمة

The Ring isomorphic
علاقة متساوية

on to
one to one
homo

ناتجة
متباينة
تشاكل

Def: A ring $(R, +, \cdot)$ is isomorphic to a ring $(\bar{R}, +, \cdot)$ if there exist $f: R \rightarrow \bar{R} \in \mathcal{F}$ is an isomorphism. R isomorphism to $\bar{R}$ denoted by $R \cong \bar{R}$

Example: A one to one mapping
 $f: R \rightarrow \bar{R}$

Sol. let $x, y \in R$
 $f(x) = f(y) \rightarrow x = y$

B, on to

Sol: let $x \in \bar{R}$, $\exists t \in R$
 $f(t) = x$

الدالة الشاملة

on to

$$f(A) = B$$

$$\text{or } R_f = B$$

الدالة متباينة

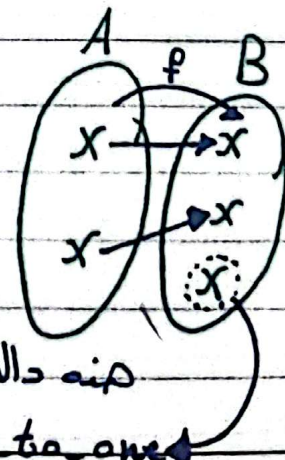
one to one

$$f(a) = f(b)$$

$$a = b$$

الدالة

$$f: A \rightarrow B$$



منه دالة

one to one

not onto

Example: let $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = 2^x, \forall x \in \mathbb{R}$ show that f is a ring isomorphism or not?

عند الفهرج تجميع الرسا

proof: 1. let $a, b \in \mathbb{R}$
 $f(a+b) = 2^{a+b} = 2^a \cdot 2^b = f(a) \cdot f(b)$
 $f(a) + f(b) = 2^a + 2^b \neq$

هذا الم يتحقق شرط

الهومو نتوقف

عن الحال

Since, $1, 2 \in \mathbb{R}$

$$f(1+2) = f(3) = 2^3 = 8$$

$$f(1) + f(2) = 2^1 + 2^2 = 2 + 4 = 6$$

$\therefore f$ is not homo

2. one to one

let $a, b \in \mathbb{R}$

$$f(a) = f(b)$$

$$2^a = 2^b \rightarrow$$

$$a = b$$

$\therefore f$ is one-to-one

عنما تتساوى

الرسا تتساوى

الرسا

3. on to

let $2^a \in \bar{\mathbb{R}}, \exists a \in \mathbb{R}$

$$\text{s.t. } f(a) = 2^a$$

إذا $R = \bar{\mathbb{R}}$

المجال يساوي المجال

المقابل إذا هو

دائماً on to

خطوات الحل:

1. نحقق homomorphism

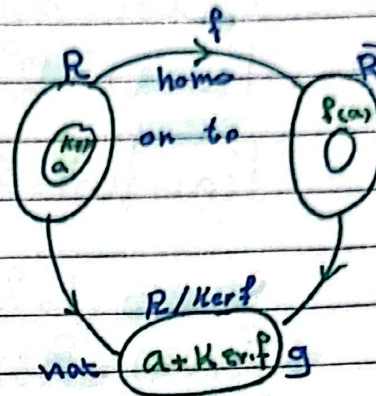
2. نحقق one-to-one

3. نحقق on to

SHAFAGH

Fundamental Homomorphism

theorem:- let f be a homo. **fundement** from the ring $(R, +, \cdot)$ on to the ring $(\bar{R}, \bar{+}, \bar{\cdot})$ then $(R/\text{Ker } f \cong \bar{R})$.



proof:- $f: (R, +, \cdot) \longrightarrow (\bar{R}, \bar{+}, \bar{\cdot})$
 be a ring homo. and **on to**.
 we must proof $(R/\text{Ker } f \cong \bar{R})$.

let $g: (R/\text{Ker } f \longrightarrow (\bar{R}, \bar{+}, \bar{\cdot}))$

$$g(a + \text{Ker } f) = f(a)$$

$$\forall a + \text{Ker } f \in R/\text{Ker } f$$

1. we proof g is homo

A. let $(a + \text{Ker } f)$

$$(b + \text{Ker } f) \mapsto G(R/\text{Ker } f)$$

نشیان گ
homo
مستند شود

الذات بـ g

1) homo

2) on to one

3) on to

$$[(a + \text{Ker } f) \oplus (b + \text{Ker } f)] = g(a + b + \text{Ker } f)$$

$$(a + b) = f(a) + f(b) = g(a + \text{Ker } f) + g(b + \text{Ker } f)$$

$$B. \quad g(a + \text{Ker } f) \otimes (b + \text{Ker } f) = g(a \cdot b + \text{Ker } f)$$

همه $f(a \cdot b) = f(a) \cdot f(b) \Rightarrow g(a + \text{Ker } f) \otimes g(b + \text{Ker } f)$

$\Rightarrow g$ is homo

2. we must proof g is one to one.

متابین
بجیب از اثبات آنها متباین

let $(a + \text{Ker } f)$
 $(b + \text{Ker } f) \mapsto \in (R / \text{Ker } f)$

$$\Rightarrow g(a + \text{Ker } f) = g(b + \text{Ker } f)$$

$$(a + \text{Ker } f) = (b + \text{Ker } f)$$

$$f(a) = f(b)$$

$$f(a) - f(b) = \bar{0}$$

$$f(a) + (-f(b)) = \bar{0}$$

$$f(a - b) = \bar{0}$$

$$\Rightarrow a - b \in \text{Ker } f$$

$$a + \text{Ker } f = b + \text{Ker } f$$

$a \in I \Rightarrow b \in I$
iff $a - b \in I$

20 3. we must proof g is onto

بجیب از اثبات آنها
مثال

Since f onto

$$\forall \bar{a} \in \bar{R}, \exists a \in R$$

$$\text{s.t. } \bar{a} = f(a)$$

$$\forall \bar{a} \in \bar{R}, \exists (a + \text{Ker } f) \in R / \text{Ker } f$$

$$\text{s.t. } g(a + \text{Ker } f) = f(a) = \bar{a}$$

$\Rightarrow g$ is onto

The Field

الحقل

Def: A non-empty set F said to be field :-

شروط الحقل

$(F, +)$ Comm. group.

$(F, \cdot)$ Comm. group.

$(\cdot)$ is dist $(+)$.

Example :- $(\mathbb{Q}, +, \cdot)$
 $(\mathbb{R}, +, \cdot)$
 $(\mathbb{C}, +, \cdot)$ Sol field

Example :- $(\mathbb{Z}, +, \cdot)$ is field ?

1, $(\mathbb{Z}, +)$ is Comm group

2, $(\mathbb{Z}, \cdot)$ is not Comm group

$\nexists a^{-1}$

$$2 \cdot 2^{-1} = 1$$

$$2 \cdot \frac{1}{2} = 1$$

$$\frac{1}{2} \notin \mathbb{Z}$$

لا تحتوي على

نظير ضربية $a \cdot a^{-1} = 1$

$\therefore (\mathbb{Z}, +, \cdot)$ is not Field

Example:- $(\mathbb{Z}_5, +, \cdot)$ is field?

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Sol

1. $(\mathbb{Z}_5, +)$ is comm. group.

$$e = a + e \Rightarrow e = 0$$

$$e = a^{-1} + a \Rightarrow 1^{-1} = 4, 0^{-1} = 0$$

$$2^{-1} = 3, 4^{-1} = 1$$

$$3^{-1} = 2$$

2. $(\mathbb{Z}_5, \cdot)$ is comm. group.

$$e = a \cdot e \Rightarrow e = 1$$

$$e = a \cdot a^{-1} \Rightarrow a \cdot a^{-1} = 1$$

$$\Rightarrow 1^{-1} = 1$$

$$2^{-1} = 3$$

$$3^{-1} = 2$$

$$4^{-1} = 4$$

3. $(\cdot)$ dist $(+)$

H.W. 1. $(\mathbb{Z}_8, +, \cdot)$ show is field or not?

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

1. $(\mathbb{Z}_8, +)$ is comm. group.

·	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	4	6	0	2	4	6
3	3	6	1	4	7	2	5
4	4	0	4	0	4	0	4
5	5	2	7	4	1	6	3
6	6	4	2	0	6	4	2
7	7	6	5	4	3	2	1

2. $(\mathbb{Z}_8, \cdot)$ is not comm. group.

لأنه لا يوجد معكوس لجميع الأعداد
مثلاً $(2, 4, 6)$ لا يوجد لها معكوس.

$\nexists a^{-1}$

$\therefore (\mathbb{Z}_8, \cdot)$ is not field.

Example: Let $F = \{a + b\sqrt{3} \mid a, b \in \mathbb{Q}\}$
 Show that $(F, +, \cdot)$ is a field?

Sol. 1. To find the identity of $(\cdot)$

Let $e = e_1 + e_2\sqrt{3}$, $e \cdot a = a$
 $\rightarrow (e_1 + e_2\sqrt{3})(a + b\sqrt{3}) = (a + b\sqrt{3})$

$\rightarrow e_1a + e_1b\sqrt{3} + e_2a\sqrt{3} + 3e_2b = a + b\sqrt{3}$
 $\rightarrow (e_1a + 3e_2b) + \sqrt{3}(e_1b + e_2a) = a + b\sqrt{3}$

$e_1a + 3e_2b = a \quad \text{--- (1)}$

$e_1b + 3e_2a = b \quad \text{--- (2)}$

من (1) نحصل على
 $e_2 = \frac{-a(-1+e_1)}{3b}$

e_2 نعوض بالمعادلة (2)

$\left(\frac{-a(-1+e_1)}{3b} \right) a + e_1b = b$

$(-1+e_1) \frac{a^2}{3b} + e_1b = b$

$(-1+e_1) \left[\frac{a^2}{3b} + b \right] = 0$

نقسم الطرفين
 بالتقسيم بالبيرو

$e_1 - 1 = 0 \rightarrow e_1 = 1 \rightarrow e_2 = 0$

$e = 1 + 0\sqrt{3}$

$$(a+b\sqrt{3})^{-1} \cdot (a+b\sqrt{3}) = (1+0\sqrt{3})$$

الحمد لله

• نظرية الحوكم
في النظم العامة

ف

$$\rightarrow \frac{a}{a^2 - 3b^2} + \frac{-b}{a^2 - 3b^2} \sqrt{3} \rightarrow (a + b\sqrt{3})^{-1}$$

$$\begin{aligned} & (a_1 + b_1\sqrt{3})(a_2 + b_2\sqrt{3}) \\ &= a_1a_2 + 3b_1b_2 + a_1b_2\sqrt{3} + a_2b_1\sqrt{3} \\ &= (a_1a_2 + 3b_1b_2) + (a_1b_2 + a_2b_1)\sqrt{3} \end{aligned}$$

Σ is closed.

Now $(F, +)$ is Comm. group.

کروڑ لاکھ

1, $(0 + \sqrt{3})$ is identity هوية

2, $(-a-b\sqrt{3})$ is inverse معکوب

$$3. (a_1 + b_1\sqrt{3}) + (a_2 + b_2\sqrt{3}) \quad \text{المجموع}$$

$$= (a_1 + a_2) + (b_1 + b_2)\sqrt{3}$$

$\therefore$ is closed.

C. Now (.) dis (+)

$$(a_1 + b_1\sqrt{3}), (a_2 + b_2\sqrt{3}), (a_3 + b_3\sqrt{3})$$

$$(a_1 + b_1\sqrt{3}) [(a_2 + b_2\sqrt{3}) + (a_3 + b_3\sqrt{3})] =$$

$$[(a_1 + b_1\sqrt{3})(a_2 + b_2\sqrt{3})] + [(a_1 + b_1\sqrt{3})(a_3 + b_3\sqrt{3})]$$

$$(a_1 + b_1\sqrt{3}) [(a_2 + b_2\sqrt{3}) + (a_3 + b_3\sqrt{3})]$$

$$(a_1 + b_1\sqrt{3}) [(a_2 + a_3) + (b_2 + b_3)\sqrt{3}]$$

نقلنا القوس على أنه
عنصر واحد

$$(a_1a_2 + a_1a_3) + (a_1b_2 + a_1b_3)\sqrt{3} + (a_2b_1 + a_3b_1)\sqrt{3}$$

$$+ (b_2b_1 + b_3b_1)3$$

الطرف اليمين

$$[(a_1 + b_1\sqrt{3})(a_2 + b_2\sqrt{3})] + [(a_1 + b_1\sqrt{3})(a_3 + b_3\sqrt{3})]$$

$$[a_1a_2 + a_1b_2\sqrt{3} + a_2b_1\sqrt{3} + b_2b_1 \cdot 3] + [a_1a_3 + a_1b_3\sqrt{3} + a_3b_1\sqrt{3} + 3b_3b_1]$$

$$(a_1a_2 + a_1a_3) + (a_1b_2 + a_1b_3)\sqrt{3} + (a_2b_1 + a_3b_1)\sqrt{3}$$

$$+ (b_2b_1 + b_3b_1)3$$

الطرف اليمين = الطرف اليمين

أسئلة مهمة عن جميع الواجبات

Ex 1- prove that $(R, +, \cdot)$ is comm ring iff $(a+b)^2 = a^2 + 2ab + b^2$?

بالقانون الحذف نبرهن الربابية .

let $(a+b)^2 = a^2 + 2ab + b^2$

$a^2 + ab + ba + b^2 = a^2 + ab + ab + b^2$

by Cancellation law (نيسط الطرفين) من قانون الحذف

$ba = ab$ (قانون الربابية بالمضرب)

let R is comm. $ab = ba$

we must prove $(a+b)^2 = a^2 + 2ab + b^2$

هنا نبرهن ان الطرفين الرئيسيين يساوي الطرفين الآخرين

$(a+b)^2 = a^2 + ab + ba + b^2$

$= a^2 + ab + ab + b^2$ (بالربابية)

$= a^2 + 2ab + b^2$ (بالثاني والمضرب)

يعتبر مبرهنة

Ex: In integral domain show that the only sol of equation $a^2 = 1$ are $a = 1$ or $a = -1$.

$a^2 = 1 \Rightarrow a^2 - 1 = 0$

فرضه $\Rightarrow (a^2 - 1)(a + 1) = 0$

اما $\Rightarrow a = 1$

أو $\Rightarrow a = -1$

كل السؤال هو عبارة عن انه ايته بالجزء تظهر اما سالبة او موجبة

Example: Let R be a ring with identity 1, has more than one element then $1 \neq 0$ ($R, \cdot$) identity $\neq 0$

proof: Since $R \neq [\emptyset]$ then there exist $a \in R, a \neq 0$
 let $1 = 0 \rightarrow$

$$a = a \cdot 1 \rightarrow a = a \cdot 0$$

$$a = 0$$

10

$$\therefore 1 \neq 0 \quad \text{C.I.}$$

Example: If $(I_1, +, \cdot)$ and $(I_2, +, \cdot)$ are ideal of the ring $(R, +, \cdot)$ s.t $I_1 \cap I_2 = [0]$, prove $a \cdot b = 0$ for every $a \in I_1$ and $b \in I_2$.

proof let $a \in I_1, b \in I_2$

Since I_1 is Ideal

$$\therefore a \cdot b \in I_1$$

Since I_2 is Ideal

$$\therefore a \cdot b \in I_2$$

$$a \cdot b \in I_1 \cap I_2$$

$$\therefore a \cdot b = 0$$

تألف مثال - Annihilator ideal $\text{ann.}(I)$

let $(I, +, \cdot)$ be an ideal at the ring $(R, +, \cdot)$ then the $\text{ann}(I)$ → اسم العنصر

$$\text{ann}(I) = \{r \in R : r \cdot a = 0, \forall a \in I\}$$

Example: let $(\mathbb{Z}_{10}, +, \cdot)$ be a ring find $\text{ann}(I)$?

$$I_1 = (2), I_2 = (5) \rightarrow$$

$$\mathbb{Z}_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$I_1 = (2) = \{0, 2, 4, 6, 8\}$$

$$\text{ann}(I_1) = (2) = \{0, 5\}$$

$$I_2 = (5) = \{0, 5\}$$

$$\text{ann}(I_2) = \{0, 2, 4, 6, 8\}$$

في $\mathbb{Z}_{10}$ عناصر I هم
تعتبر المدورة.

الاعطية بالسؤال
تأتي من البصيرة
التي هي $\mathbb{Z}_{10}$
2x5

نأخذ $\mathbb{Z}_{10}$ من
ونفرضه كل عناصر I
إذا ظهر صفر إذن
هو المدور
5 نفرضه بالعناصر I
فسوف تظهر
تحت الخيار ١٥

ample: let $(\mathbb{Z}_8, +, \cdot)$ be a ring find $\text{ann}(I)$?

$$\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

$$I_1 = (2) = \{0, 2, 4, 6\}$$

$$\text{ann}(I_1) = \{0, 4\}$$

$$I_2 = (4) = \{0, 4\}$$

$$\text{ann}(I_2) = \{0, 2, 4, 6\}$$

Example 1: let $(Z_{12}, +, \cdot)$ be a ring find $\text{ann}(I)$?

Sol: $Z_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ مجموعة فعلية 2×6

$$I_1 = (2) = \{0, 2, 4, 6, 8, 10\}$$

$$\text{ann}(2) = \{0, 6\}$$

$$I_2 = (3) = \{0, 3, 6, 9\}$$

$$\text{ann}(3) = \{0, 4, 8\}$$

$$I_3 = (4) = \{0, 4, 8\}$$

$$\text{ann}(4) = \{0, 3, 6, 9\}$$

$$I_4 = (6) = \{0, 6\}$$

$$\text{ann}(6) = \{0, 2, 4, 6, 8, 10\}$$

$$6 \times 0 = 0$$

$$6 \times 2 = 12 = 0$$

$$6 \times 4 = 24 = 0$$

$$6 \times 6 = 36 = 0$$

$$6 \times 8 = 48 = 0$$

$$6 \times 10 = 60 = 0$$

Example 2: let $(Z_{30}, +, \cdot)$ be a ring find $\text{ann}(I)$?

Sol: $Z_{30} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29\}$

5×6

2×15

3×10

$$I_1 = (2) = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28\}$$

$$\text{ann}(2) = \{0, 15\}$$

$$I_2 = (3) = \{0, 6, 9, 12, 15, 18, 21, 24, 27\}$$

$$\text{ann}(3) = \{0, 10, 20\}$$

$$I_3 = (5) = \{0, 5, 10, 15, 20, 25\}$$

$$\text{ann}(5) = \{0, 6, 12, 18, 24\}$$

$$I_4 = (6) = \{0, 6, 12, 18, 24\}$$

$$\text{ann}(6) = \{0, 5, 10, 15, 20, 25\}$$

$$I_5 = (10) = \{0, 10, 20\}$$

$$\text{ann}(10) = \{0, 6, 9, 12, 15, 18, 21, 24, 27\}$$

SHAFAGH $I_6 = (15) = \{0, 15\}$

$$\text{ann}(15) = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28\}$$



السبع يكملو

Theorem: Every field is an integral domain.

proof: let $(F, +, \cdot)$ is field and
let $a, b \in F$
s.t. $a \cdot b = 0$

1. if $a \neq 0, a \in F$

Since F is field

$\therefore \exists a^{-1} \in F$

$$a^{-1} (a \cdot b = 0)$$

$$(a^{-1} \cdot a) b = a^{-1} \cdot 0$$

$$1 \cdot b = 0$$

$$b = 0$$

2. if $b \neq 0, b \in F$

$\therefore \exists b^{-1} \in F$

$$b^{-1} (a \cdot b = 0)$$

$$a (b \cdot b^{-1}) = 0 \cdot b^{-1}$$

$$a \cdot 1 = 0$$

$$a = 0$$

$\therefore F$ is integral domain

تعتبر حقلًا.

Sub Field - حقل جزئي

Def:

A sub $A \neq \emptyset$ of field F is said to be sub field if

شروط الحقل الجزئي

- 1, $A \neq \emptyset$
- 2, $a, b \in A, \forall a, b \in A$
- 3, $a, b^{-1} \in A, \forall a, b \in A$

Example: $(\mathbb{Q}, +, \cdot)$ is sub field of $(\mathbb{R}, +, \cdot)$?

Sol, 1, $\mathbb{Q} \neq \emptyset$

2, $a, b \in \mathbb{Q}, \forall a, b \in \mathbb{Q}$

$$a = \frac{1}{2}, b = \frac{1}{3}$$

$$\frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6} \in \mathbb{Q}$$

3, $a, b^{-1} \in \mathbb{Q}, \forall a, b \in \mathbb{Q}$

$$a = \frac{1}{2}, b^{-1} = 3$$

$$b = \left(\frac{1}{3}\right)^{-1} = 3$$

$$\frac{1}{2} \cdot 3 = \frac{3}{2} \in \mathbb{Q}$$

$\therefore (\mathbb{Q}, +, \cdot)$ is sub field.

Q

IAFAGH

Example: Let $S = \{a+b\sqrt{2} : a, b \in \mathbb{Q}\}$ is not empty set of field $(\mathbb{R}, +, \cdot)$ is empty set of field $(\mathbb{R}, +, \cdot)$ or $(\mathbb{R}, +, \cdot)$ on subfield of $(\mathbb{R}, +, \cdot)$ or not?

Sol. $\forall (a_1+b_1\sqrt{2}), (a_2+b_2\sqrt{2}) \in S$ محقق شرط حقل

$a-b \in S, \forall a, b \in S$

الشرط الأول

$$\begin{aligned} (a_1+b_1\sqrt{2}) - (a_2+b_2\sqrt{2}) &\in S \\ (a_1-a_2) + (b_1-b_2)\sqrt{2} &\in S \\ (a_1-a_2), (b_1-b_2) &\in \mathbb{Q} \end{aligned}$$

الشرط الثاني

$a \cdot b^{-1} \in S, \forall a, b \in S$
 هنا نجد في المعكوس $(a_1+b_1\sqrt{2})^{-1} = \frac{1}{a_1+b_1\sqrt{2}}$ من خلال قانون

$$\frac{1}{a_1+b_1\sqrt{2}} = \frac{a_1-b_1\sqrt{2}}{(a_1+b_1\sqrt{2})(a_1-b_1\sqrt{2})} = \frac{a_1-b_1\sqrt{2}}{a_1^2-b_1^2 \cdot 2}$$

$$(a_2+b_2\sqrt{2})^{-1} = \frac{1}{a_2+b_2\sqrt{2}} \cdot \frac{a_2-b_2\sqrt{2}}{a_2-b_2\sqrt{2}} = \frac{a_2-b_2\sqrt{2}}{a_2^2-b_2^2 \cdot 2}$$

$$\frac{a_2-b_2\sqrt{2}}{a_2^2-b_2^2 \cdot 2} \cdot \frac{a_1-b_1\sqrt{2}}{a_1^2-b_1^2 \cdot 2} = \frac{(a_2-b_2\sqrt{2})(a_1-b_1\sqrt{2})}{(a_2^2-b_2^2 \cdot 2)(a_1^2-b_1^2 \cdot 2)}$$

$$= \frac{a_2-b_2\sqrt{2}}{a_2^2-b_2^2 \cdot 2} \cdot \frac{a_1-b_1\sqrt{2}}{a_1^2-b_1^2 \cdot 2}$$

في كل من a_1, a_2 محقق شرط حقل

$$= (a_1+b_1\sqrt{2}) \cdot \frac{a_2-b_2\sqrt{2}}{a_2^2-b_2^2 \cdot 2}$$

ونضع المقام على حدود البسط

$$(a_1+b_1\sqrt{2}) \left[\frac{a_2}{a_2^2-b_2^2 \cdot 2} + \frac{-b_2\sqrt{2}}{a_2^2-b_2^2 \cdot 2} \right] \rightarrow \frac{a_1 a_2 - b_1 b_2 \cdot 2}{a_2^2-b_2^2 \cdot 2}$$

SHAFAH

التوزيع مع تكافؤ الحل $\rightarrow$ يسج

$$\frac{a_1 a_2}{a_1^2 - 2b_2^2} + \frac{-2b_1 b_2}{a_1^2 - 2b_2^2} + \left(\frac{b_1 a_2}{a_1^2 - 2b_2^2} + \frac{-b_2 a_1}{a_1^2 - 2b_2^2} \right)$$

التوسيع القوس الأول كيف ظهر

$$\frac{a_2}{a_1^2 - 2b_2^2} + \frac{-b_2}{a_1^2 - 2b_2^2} \sqrt{2}$$

$$\frac{a_1 a_2}{a_1^2 - 2b_2^2} + \frac{-2b_1 b_2}{a_1^2 - 2b_2^2} + \left(\frac{b_1 a_2}{a_1^2 - 2b_2^2} + \frac{-b_2 a_1}{a_1^2 - 2b_2^2} \right)$$

قوس
الوسط

الوسط والوسط
الثاني والثاني

$$\frac{a_2}{a_1^2 - 2b_2^2} + \frac{-b_2}{a_1^2 - 2b_2^2} \sqrt{2}$$

توسيع القوس الثاني

الوسط والخارج

خارج

$$\frac{a_2}{a_1^2 - 2b_2^2} + \frac{-a_1 b_2}{a_1^2 - 2b_2^2} \sqrt{2}$$

سحب الجذر
عامل مشترك

القوس الثاني

$$\frac{b_1 a_2}{a_1^2 - 2b_2^2} + \frac{-a_1 b_2}{a_1^2 - 2b_2^2} \sqrt{2}$$

التوسيع

Simple ring

حلقه بسيطة

Def:- A ring $(R, +, \cdot)$ is said to be Simple ring if it has no proper ideal.

الحلقة بسيطة اذا كانت لا تحتوي على اي مثالي صحيح.

Example:- Show that $(\mathbb{Q}, +, \cdot)$ is a Simple ring.

proof:- Let $(I, +, \cdot)$ be any ideal of $(\mathbb{Q}, +, \cdot)$

let $a \in I, a \neq 0, a \in \mathbb{Q}$

Since $\mathbb{Q}$ is field $\therefore 1 \in \mathbb{Q}$

$\exists a^{-1} \in \mathbb{Q}$ s.t. $a^{-1} \cdot a = 1 \in I$

$I \subseteq \mathbb{Q}$

$1 \in I$

$I \subseteq \mathbb{Q}$

$\therefore I$ is Simple.

The types of ideals

The maximal ideal

مثالي اعظم

Def: let $(R, +, \cdot)$ be a comm. ring with identity and $(I, +, \cdot)$ is an ideal of the ring R is said to be maximal ideal $I \neq R$ and whenever $(J, +, \cdot)$ is an ideal of R with $I \subset J \subseteq R \rightarrow J = R$.

Q/ give me example for how maximal?

Example: $(\mathbb{Z}_{12}, +, \cdot)$ Find all maximal ideal?

sol, $\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

$$(2) = \{0, 2, 4, 6, 8, 10\} \text{ max}$$

بأكثر مجموعة

$$(4) = \{0, 4, 8\}$$

تحتوي على عناصر

$$(3) = \{0, 3, 6, 9\} \text{ max}$$

هذه max

$$(6) = \{0, 6\}$$

$\therefore (2), (3)$ how max.

Example: $(\mathbb{Z}_8, +, \cdot)$ Find all max?

Sol

$$(2) = \{0, 2, 4, 6\} \text{ max.}$$

$$(4) = \{0, 4\}$$

AGH

Theorem: let $(R, +, \cdot)$ be a comm ring with identity and M is a proper ideal of R . then M is maximal ideal iff (R/M) is field. $\rightarrow$ (x و y) فراغات

بدون
فقط
بدون

Example: $R = (\mathbb{Z}, +, \cdot)$ مثال برای هر گشت
Prove that $\mathbb{Z}/(3)$ is field?

Sol

$$(3) = M = \{ 0, \pm 3, \pm 6, \pm 9, \dots \}$$

$$\mathbb{Z}/(3) = [0 + (3), 1 + (3), 2 + (3)] \cong \mathbb{Z}_3$$

Now prove $\mathbb{Z}/(3)$ is field?

1. $(\mathbb{Z}/(3), +)$ is Comm. group.

+	$0 + (3)$	$1 + (3)$	$2 + (3)$
$0 + (3)$	$0 + (3)$	$1 + (3)$	$2 + (3)$
$1 + (3)$	$1 + (3)$	$2 + (3)$	$0 + (3)$
$2 + (3)$	$2 + (3)$	$0 + (3)$	$1 + (3)$

2. $(\mathbb{Z}/(3), \cdot)$ is Comm. group.

$\cdot$	$0 + (3)$	$1 + (3)$	$2 + (3)$
$0 + (3)$	$0 + (3)$	$0 + (3)$	$0 + (3)$
$1 + (3)$	$0 + (3)$	$1 + (3)$	$2 + (3)$
$2 + (3)$	$0 + (3)$	$2 + (3)$	$1 + (3)$

3. $(\cdot)$ dist $(+)$.

SHAFAGH
تبع

$$0 + (3) [1 + (3) + 2 + (3)] = [0 + (3) \cdot 1 + (3)] + [0 + (3) \cdot 2 + (3)]$$

$$0 + (3) \cdot 0 + (3) = 0 + (3) + 0 + (3)$$

$$0 + (3) = 0 + (3)$$

Lemma 1: An element is invertible if it belongs to no maximal ideal.

المركب غير في المثالي الرعظم ليس له معكوس عكسي

give example An element invertible

Example: $(\mathbb{Z}_{12}, +, \cdot)$ مثال على غير مركبة

Sol $\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$
 $(2) = \{0, 2, 4, 6, 8, 10\}$ max ideal.

$$2 \cdot \boxed{} = 1$$

$2 \in \text{max}$ ليس لها معكوس عكسي

$$4 \cdot \boxed{} = 1$$

Lemma 2: the only maximal ideal in the field F is the Zero ideal.

Proof: let $I \neq [0]$ be maximal ideal in F let $0 \neq a \in I$

$a \in F, a^{-1} \in F$ [Since F is Field by lemma 1: [An element is ...]]

Example: $(\mathbb{Z}_5, +, \cdot)$

$$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$$

$$1 \cdot \boxed{1} = 1$$

$$2 \cdot \boxed{3} = 1$$

$$3 \cdot \boxed{2} = 1$$

$$4 \cdot \boxed{4} = 1$$

Prime Ideal...

Def:- An ideal I of the ring R is said to be prime Ideal if for all $a, b \in R$, $a \cdot b \in I$ then $a \in I$ or $b \in I$.

ايديال I مثاليه اوليه اذا كان حاصل ضرب كل العناصر تنتمي الى I ويكون اما a او b ينتمي الى I .

Example:- $R = (\mathbb{Z}, +, \cdot)$ Find prim Ideal?

$$I = (2) = \{0, \pm 2, \pm 4, \pm 6, \pm 8, \dots\}$$

		$b \in I$	$a \notin I$
Sol	$\pm 2 \rightarrow 1 \cdot 2$	$2 \in I$	$1 \notin I$
	$\pm 4 \rightarrow 1 \cdot 4$	$4 \in I$	$1 \notin I$
	$\pm 6 \rightarrow 2 \cdot 3$	$2 \in I$	$3 \notin I$

$\therefore I = \{0, \pm 2, \pm 4, \pm 6, \pm 8, \dots\}$ prime $\rightarrow$ تاخذ ارقام التي لرتبتي

Not 1. every maximal ideal is prime ideal

but the convers is not true.

كله مثاليه اعظم هو مثاليه اوليه والعكس غير صحيح.

H.w / give example maximal ideal is prime ideal?

Sol

Lemma 1. An Comm ring with identity is an integral domain iff the zero ideal is a prime ideal.

Example 1. $\mathbb{Z}_5$ - integral
→ proper (0).

Theorem 1. let I be a proper ideal of the Comm ring with identity R , then R/I is an integral domain iff I is a prime Ideal.

proof: let I be a prime ideal of R
Since R is a Comm ring with identity so R/I

Now we must prove that R/I with out zero divisor.
as Summe that $(a+I)(b+I) = 0+I = I$ $a, b \in R$

$$\rightarrow (ab+I) = I \rightarrow ab \in I$$

$\because I$ is prime $\rightarrow$ either $a \in I$ or $b \in I$

$$\rightarrow a+I = I \text{ or } b+I = I$$

$\therefore R/I$ is with out zero divisor.

$\therefore R/I$ is integral domain.

→ Suppose that R/I integral domain
and let $ab \in I$

→ $(a+I)(b+I) = I = 0+I$
⇒ R/I has no zero divisor

→ $a+I = I$ or $b+I = I$
→ $a \in I$ or $b \in I$

⇒ I is a prime Ideal...

Ideal Generated by the Set

المثال الذي يتولد بالجموعة

Def: let $(I, +, \cdot)$ an ideal of comm ring with identity $(R, +, \cdot)$ and $a \in R, a \notin I$, then $((I, a), +, \cdot)$ is ideal generated by the set $I \cup \{a\}$ s.t. $I \subset (I, a) \subseteq R$

I مثال مع حلقة ابدال وعناصر حايه ... يوجد عنصر

$a \in R, a \notin I$ يتولد لدينا زوج مرتب (I, a)

حيث انه $a \in R, i \in I$

Example: $(\mathbb{Z}_{15}, +, \cdot)$ is ring $I_1 = (3), I_2 = (5)$

Find: $((3), 4), +, \cdot$ و $((5), 4), +, \cdot$?

$\mathbb{Z}_{15} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$

$I_1 = (3) = \{0, 3, 6, 9, 12\}$

$I_2 = (5) = \{0, 5, 10\}$ نأخذ اول زمين I ونجعله مع (4) من R

ونفرضه بال ارقام $\mathbb{Z}_{15}$ $(4, 3) = \{i + ar\}$ $i \in I, r \in R$

$0 + (4) \cdot 1 = 4$	$(3), 4, +, \cdot$	$12 + (4) \cdot 1 = 16$
$0 + (4) \cdot 2 = 8$		$12 + (4) \cdot 2 = 20$
$\vdots$	$(5), 4, +, \cdot$	$\vdots$
$0 + (4) \cdot 14 = 56$		$12 + (4) \cdot 14 = 68$
$3 + (4) \cdot 1 = 7$	$6 + (4) \cdot 1 = 10$	$9 + (4) \cdot 1 = 13$
$3 + (4) \cdot 2 = 11$	$6 + (4) \cdot 2 = 14$	$9 + (4) \cdot 2 = 17$
$\vdots$	$\vdots$	$\vdots$
$3 + (4) \cdot 14 = 59$	$6 + (4) \cdot 14 = 62$	$9 + (4) \cdot 14 = 65$

Theorem: let R be Comm. ring with identity
 I is ideal prove (I, a) is an Ideal

proof, 1. $(I, a) \neq \emptyset$

Since $0 \in I \rightarrow 0 \in (I, a)$

2. let $i_1 + ar_1$ and $i_2 + ar_2 \in (I, a)$
 $i_1, i_2 \in I, a \notin I$

3. $(I, a) = \{i + ar : i \in I, r \in R, a \notin I\}$

$$(i_1 + ar_1) - (i_2 + ar_2) = (i_1 - i_2) + a(r_1 - r_2)$$

$i_1 - i_2 \in I$ $a(r_1 - r_2) \in (I, a)$

3. $\forall r \in R, i + ar \in (I, a)$

$$r(i + ar)$$

$$= ir + rar \in (I, a)$$

$\therefore$ we get (I, a) is Ideal.

مثالب إثباته primary ideal

1	2	3	4	5	6	7	8	9	10
Mo	Tu	We	Th	Fr	Sa	Su	Mo	Tu	We

Def: An ideal I of the ring R is said to be primary ideal if $a, b \in R$ and $a \cdot b \in I$ with $a \notin I$ Then $b^n \in I, n \in \mathbb{Z}^+$

I مثالب إثباته يوجد عنصران في R (a, b) ناتج $a \cdot b$ ينتمي إلى I إذا $a \notin I$ كان $b^n \in I$ وكانت n أكبر من صفر موجب.

example: $(\mathbb{Z}, +, \cdot)$ is a ring $(n), +, \cdot$ ideal in $\mathbb{Z}$
 $n \in \mathbb{Z}^+$ Find or $(2), +, \cdot$ is primary ideal? **هل المثالب (2) هو مثالب إثباته؟**

$$\mathbb{Z} = \{0, 1, 2, \dots\}$$

$$I = (2) = \{0, 2, 4, 6, \dots\} \text{ max} \rightarrow \text{primary}$$

$$\{1, 2 \in \mathbb{Z} \rightarrow 1 \cdot 2 = 2 \in I \rightarrow 1 \notin I, 2 \in I\}$$

$$\text{عنصر } a \notin I \quad \text{ناتج ضربهما } \in I \quad \text{عنصر } b \in I$$

$\times$ نختار عنصرين ينتمون إلى $\mathbb{Z}$ بشرط العنصر الأول لا ينتمي إلى I والعنصر الثاني ينتمي إلى I مع رفعه لـ n موجب.

example: $(\mathbb{Z}, +, \cdot)$ is a ring $(3), +, \cdot$ is primary ideal

$$\mathbb{Z} = \{0, 1, 2, 3, \dots\}$$

$$I = (3) = \{0, 3, 6, 9, \dots\}$$

$$\{1, 6 \in \mathbb{Z} \rightarrow 1 \cdot 6 = 6 \in I \rightarrow 1 \notin I, 6 \in I\}$$

Example. $(\mathbb{Z}, +, \cdot)$ is a ring $\rightarrow ((5), +, \cdot)$ is primary?

Sol. $\mathbb{Z} = \{0, 1, 2, 3, \dots\}$
 $I = (5) = \{0, 5, 10, 15, \dots\}$

$\exists 2, 5 \in \mathbb{Z} \rightarrow 2 \cdot 5 = 10 \in I \rightarrow 2 \notin I, 5 \in I$
 $\therefore$ primary.

Example. $(\mathbb{Z}, +, \cdot)$ is a ring $\rightarrow ((7), +, \cdot)$ is primary?

Sol. $\mathbb{Z} = \{0, 1, 2, 3, \dots\}$
 $I = (7) = \{0, 7, 14, 21, \dots\}$

$\exists 3, 7 \in \mathbb{Z} \rightarrow 3 \cdot 7 = 21 \in I \rightarrow 3 \notin I, 7 \in I$
 $\therefore$ primary.

Example. $(\mathbb{Z}, +, \cdot)$ is a ring $\rightarrow ((9), +, \cdot)$ is primary?

Sol. $\mathbb{Z} = \{0, 1, 2, 3, \dots\}$
 $I = (9) = \{0, 9, 18, 27, \dots\}$

$\exists 1, 9 \in \mathbb{Z} \rightarrow 1 \cdot 9 = 9 \in I \rightarrow 1 \notin I, 9 \in I$
 $\therefore$ primary.

* Every prime is primary

ہر اولیٰ ہو آبائی

Max $\rightarrow$ prime $\rightarrow$ primary

SHAFAGH

Max $\rightarrow$ prime $\rightarrow$ primary

give example of primary ideal which is not prime ideal.

The ring $(\mathbb{Z}_{12}, +, \cdot)$ an ideal $I = (4) = \{0, 4, 8\}$

اعتبر مثال مثالي ابتدائي لكنه ليس مثالي أولي

Lemma: Every prime ideal is primary ideal

Proof: let I be a prime ideal and let

$a, b \in R, a \notin I, b \in I, a \cdot b \in I$

[by Def prime]

$\rightarrow b^n \in I \rightarrow I$ is primary

جاكوبسون

Def: The radical of a ring R (rad) is the set $J(R) = \text{Rad}(R)$.

$$J(R) = \text{Rad}(R) = \bigcap \{ M : M \text{ is maximal ideal in } R \}$$

موتلاع كل امثاليات العنصر في الحلقه R

Note:

① If $J(R) = \{0\}$ then R is said to be semi-simple ring.

② $J(R)$ is ideal.

③ every ring has Jacobson at least $J(R) = \{0\}$.

Example: $(\mathbb{Z}_{12}, +, \cdot)$ is ring. $M_1 = (2), M_2 = (3)$
Find $J(\mathbb{Z}_{12})$?

Soln

$$M_1 = (2) = \{0, 2, 4, 6, 8, 10\}$$

$$M_2 = (3) = \{0, 3, 6, 9\}$$

$$M_1 \cap M_2 = \{0, 6\}$$

$$\therefore J(\mathbb{Z}_{12}) = \{0, 6\}$$

3x4
2x6
prime $\mathbb{Z}_{12}/(2), (3)$

ماتر النظام الروايه فقط

Example: $(\mathbb{Z}_{15}, +, \cdot)$ is ring. Find $\text{rad}(\mathbb{Z}_{15})$?

$$M_1 = (3) = \{0, 3, 6, 9, 12\}$$

$$M_2 = (5) = \{0, 5, 10\}$$

$$M_1 \cap M_2 = \{0\}$$

$$\therefore J(\mathbb{Z}_{15}) = \{0\} \text{ simple ring}$$

Example: $(\mathbb{Z}_{20}, +, \cdot)$ is ring. Find $J(\mathbb{Z}_{20})$?

$$M_1 = (2) = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$$

$$M_2 = (5) = \{0, 5, 10, 15\}$$

$$M_1 \cap M_2 = \{0, 10\}$$

$$= J(\mathbb{Z}_{20}) = \{0, 10\}$$

4 x 5

2 x 10

prime (2), (5)

ناقصات الاولى

Subject: _____ Date: _____
 Lemma: The only idempotent elements in $J(R)$ is zero.

Prove: let $a \in J(R)$
 s.t. $a^2 = a \rightarrow a^2 - a = 0$

by lemma 1:-

[let R be ring with identity then the element $a \in J(R)$ iff the element $1 - ra$ has inverse for all $r \in R$]

$\therefore 1 - ar$ has inverse

iff $r = 1$

$\therefore (1 - a)b = 1$ inverse

$$a(1 - a)b = a \cdot 1$$

$$\rightarrow (a - a^2)b = a$$

$$\rightarrow a = 0$$

Q/give example The only idempotent element in $J(R)$ is zero?

Sol in $R = (\mathbb{Z}_{12})$ Max (2, 13)

$$J(R) = J(\mathbb{Z}_{12}) = [0, 6] \rightarrow 2, 3 \text{ Anti}$$

$$0^2 = 0$$

$$6^2 \neq 6$$

$\therefore 0$ is only idempotent element

SHAFAGH

Theorem 1. Let I be an ideal of the ring R , then $I \subseteq J(R)$ iff every element of the coset $1+I$ has an inverse in R .

proof: Suppose that $I \subseteq J(R)$ and there exist $\exists a \in I$ s.t. $1+a$ has no inverse. by

[the element a in a comm. ring with identity has inverse iff a is not belong to maximal ideal of R].

→ $\exists M$ maximal ideal s.t. $1+a \in M$.

$a \in I \subseteq J(R) \rightarrow a \in J(R)$

→ $a \in M$

$1+a \in M$

→ $1 \in M = R$ C!

$a \in M, 1 \in M$

← assume that every element of the coset $1+I$ has inverse let $I \subseteq J(R)$

→ $\exists$ a maximal ideal M of R

s.t. $I \not\subseteq M$. if a is any element I and $a \notin M$.

Subject: _____
 Year: _____
 Month: _____
 Date: _____

Sa	Su	Mo	Tu	We	Th	Fr
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$$(M, a) = \{m + ra, r \in R, m \in M, a \notin M\} = R$$

$$m + ra = I, I \in R$$

$$I - ra = m \rightarrow (I - (-ar)) = m \in M$$

$$I + (-ra) \in I + I = m$$

$$I + (-ra) \in I + I = m \in M$$

C!

كل عنصر في الرعد هو له اريحتوى

على فاكوس

الجزء الأول Nil radical $\sqrt{I}$

Def: let I be an ideal of the ring R .
 the nil radical of I is the set

$$\sqrt{I} = \{r \in R : r^n \in I, n \in \mathbb{Z}^+\}$$

Example: $R = (\mathbb{Z}_8, +, \cdot)$ Find $\sqrt{(2)}, \sqrt{(3)}, \sqrt{(6)}, \sqrt{(9)}$?

Sol: $\sqrt{(2)} = (2) = \{0, 2, 4, 6, 8, 10, 12, 14, 16\}$
 $0^2 = 0 \in \sqrt{I}$
 $2^2 = 4 \in \sqrt{I}$
 $\vdots$
 $16^2 \text{ ? } \in \sqrt{I}$

$$\sqrt{(3)} = (3) = \{0, 3, 6, 9, 12, 15\}$$

$$\sqrt{(6)} = \sqrt{2 \cdot 3} = (6) = \{0, 6, 12\}$$

$$\sqrt{(9)} = \sqrt{3^2} = (3) = \{0, 3, 6, 9, 12, 15\}$$

$$\sqrt{p^2} = p \rightarrow \text{قائم} \quad \sqrt{9} = \sqrt{3^2} = 3$$

$$\sqrt{pq} = pq \rightarrow \text{قائم} \quad \sqrt{6} = \sqrt{2 \cdot 3} = 6$$

Subject _____ Date _____
H.W. in ring of $(\mathbb{Z}_{30}, +, \cdot)$ find $\sqrt{(3)}$, $\sqrt{(5)}$.

Sol. $\sqrt{3} = (3) = \{0, 3, 6, 9, 12, 15, 18, 21, 24, 27\}$

$\sqrt{(5)} = (5) = \{0, 5, 10, 15, 20, 25\}$

$\sqrt{15} = \sqrt{(3 \cdot 5)} = (15) = \{0, 15\}$

أبني حياة الإنسان وفادوسع رحمة الله

حلقة أولية Prime ring

Def:- let $(R, +, \cdot)$ be a ring, then R is said to be semi-prime if $\sqrt{I} = I$ for all ideal.

Example 1- $R = (\mathbb{Z}_{15}, +, \cdot)$ is for semi-prime

Sol

$$I = (3) = \{0, 3, 6, 9, 12\}$$

$$\sqrt{(3)} = (3)$$

$$I = (5) = \{0, 5, 10\}$$

$$\sqrt{(5)} = (5)$$

$\therefore \mathbb{Z}_{15}$ is semi-prime ring

Example 2- $R = (\mathbb{Z}_{18}, +, \cdot)$

Sol

$$\sqrt{(2)} = (2) = \{0, 2, 4, 6, 8, 10, 12, 14, 16\}$$

$$\sqrt{(3)} = (3) = \{0, 3, 6, 9, 12, 15\}$$

$$\sqrt{(6)} = (6) = \{0, 6, 12\}$$

$$\sqrt{(9)} = (9)$$

* اي رقم يتجزأ لـ 9 يساوي حلقة اولية مثلاً $\sqrt{9} = 9$

Theorem: let $(R, +, \cdot)$ be a ring and let I be any ideal of R , then $\sqrt{I}$ is an ideal of R and $I \subseteq \sqrt{I}$.

proof: let we prove $\sqrt{I}$ is ideal

1. $\sqrt{I} \neq \emptyset$ since $0 \in \sqrt{I}$ i.e. $0^n \in I$

2. let $a, b \in \sqrt{I} \rightarrow a^n \in I, b^n \in I$

we must prove that $a - b \in \sqrt{I}$

~~$(a-b)^n \in I$ يجب ان نشبه ان~~

$$(a-b)^n = a^n - n \frac{a^{n-1}b}{1} + \frac{n(n-1)a^{n-2}b^2}{2!} + \dots + b^n \in I$$

Since $(a-b)^n \in I \rightarrow a-b \in \sqrt{I}$

3. $\forall r \in R, a \in \sqrt{I}$

$$(ar)^n = a^n r^n \in I \rightarrow ar \in \sqrt{I}$$

$$(ra)^n = r^n a^n \in I \rightarrow ra \in \sqrt{I}$$

$\therefore \sqrt{I}$ is ideal.

Theorem: let R be a Comm ring with identity and let I be a prime ideal of R then I is primary ideal iff every zero divisor in R/I is nilpotent

proof: Supp that I is primary ideal of R . and let $x+I \in R/I$ is zero divisor

$$\exists y+I \in R/I$$

$$\text{s.t. } (x+I)(y+I) = 0+I$$

$$\text{and } y+I \neq I$$

$$xy+I = I \text{ and } y+I \neq I$$

$$xy \in I \text{ and } y \notin I$$

$$\therefore I \text{ is primary ideal [a,b} \in I, a \notin I, b^n \in I]$$

$$\therefore x^n \in I \rightarrow x^n + I = I$$

$$(x+I)^n = I$$

$\therefore x+I$ is nilpotent element in R/I

توضيح

$$y+I \neq I+0$$

$$a \cdot b = 0$$

شرط البرهان

توضيح

$$a^n = 0 \text{ nil}$$

$$(x+I)^n = I (RA)$$

$$x+I \text{ nilpot}$$

→ assume that every zero divisor of R/I is nilpotent in R/I
 we must prove I is primary

$$[ab \in I, a \notin I, b^n \in I]$$

$$\text{let } x, y \in R, xy \in I, y \notin I$$

$$\rightarrow x^n \in I$$

$$(x+I)(y+I) = I, y+I \neq I$$

$$(x+I)^n = I \text{ [every zero divisor of } R/I \text{ is nilpotent]}$$

$$x^n + I = I$$

$$\rightarrow x^n \in I$$

$$\rightarrow I \text{ is primary.}$$

$$\text{we must prove } I \subseteq \sqrt{I}$$

$$\text{let } a \in I \rightarrow a^2 \in I$$

$$\rightarrow a^3 \in I \dots a^n \in I$$

$$\therefore a \in \sqrt{I} \rightarrow I \subseteq \sqrt{I}$$

Theorem: prove that $I \subseteq \sqrt{I}$ by ex. in $\mathbb{Z}_{18}$

proof, $(9) = [0, 9]$

ex

SHAFAGH

$$\sqrt{9} = \sqrt{3^2} = (3) = \{0, 3, 6, 9, 12, 15\}$$

Theorem: If I and J are ideal of ring R then:-

1. $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$ المبرهنة الأولى

Proof: let $x \in \sqrt{I \cap J}$

$$\iff x^n \in I \cap J$$

$$\iff x^n \in I \cap J, n \in \mathbb{Z}^+$$

$$\iff x^n \in I \text{ and } x^n \in J$$

$$\iff x \in \sqrt{I} \text{ and } x \in \sqrt{J}$$

$$\iff x \in \sqrt{I} \cap \sqrt{J}$$

2. $\sqrt{I+J} = \sqrt{I} + \sqrt{J}$ المبرهنة الثانية

Proof: let $x \in \sqrt{I} + \sqrt{J}$

where $x = a + b$, $a \in \sqrt{I}$, $b \in \sqrt{J}$

$$\rightarrow m \neq n \in \mathbb{Z}^+, a^m \in I, b^n \in J$$

then $(a+b)^{n+m} = a^{n+m} + (n+m)a^{n+m-1}b + \dots + b^{n+m}$
 $= a^n \cdot a^m + (n+m)a^n \cdot a^{m-1}b + \dots + b^n b^m \in I+J$

$$\therefore (a+b)^{n+m} \in I+J$$

$$\rightarrow a+b \in \sqrt{I+J}$$

$$\therefore x \in \sqrt{I+J}$$

$$\rightarrow \sqrt{I} + \sqrt{J} \subseteq \sqrt{I+J}$$

3. $\sqrt{\sqrt{I}} = \sqrt{I}$

للمبرهنة الثالثة

Proof: let $x \in \sqrt{\sqrt{I}} \leftrightarrow x^n \in \sqrt{I}, n \in \mathbb{Z}^+$
 if $(x^n)^m \in I, m \in \mathbb{Z}^+$

$$x^r \in I [r = n \cdot m] \rightarrow x \in \sqrt{I}$$

4. I is prime $\rightarrow I = \sqrt{I}$

للمبرهنة الرابعة

Proof: let $a \in \sqrt{I} \rightarrow a^n \in I, n \in \mathbb{Z}^+$

$$\rightarrow a \in I [\text{Since } I \text{ is an ideal}]$$

$$\therefore \sqrt{I} \subseteq I \text{ --- } \textcircled{\sigma}$$

Now let $a \in I \rightarrow a^n \in I [I \text{ is an ideal}]$
 $\therefore a \in \sqrt{I} \rightarrow I \subseteq \sqrt{I} \text{ --- } \textcircled{\text{2}}$

From $\textcircled{\sigma}$ and $\textcircled{\text{2}}$ we get $I = \sqrt{I}$

5. if $I \subseteq J \subseteq \sqrt{I} \rightarrow \sqrt{I} = \sqrt{J}$

للمبرهنة الخامسة

Proof: let $a \in \sqrt{I} \rightarrow a^n \in I, n \in \mathbb{Z}^+$

$$\text{Since } I \subseteq J \rightarrow a^n \in J \rightarrow a \in \sqrt{J}$$

$$\therefore \sqrt{I} \subseteq \sqrt{J} \text{ --- } \textcircled{\sigma}$$

$$\text{let } a \in \sqrt{J} \rightarrow a^n \in J \rightarrow a^n \in \sqrt{I}$$

$$(a^n)^m \in I \rightarrow a^r \in I, r \in \mathbb{Z}^+ \rightarrow a \in \sqrt{I}$$

$$\therefore \sqrt{J} \subseteq \sqrt{I} \text{ --- } \textcircled{\text{2}}$$

From $\textcircled{\sigma}$ and $\textcircled{\text{2}}$ we get $\sqrt{I} = \sqrt{J}$

SHAFAGH

Regular ring

Def: A ring R is said to be regular ring if for each $a \in R$, $\exists b \in R$ such that $a = a \cdot b \cdot a$.
مع أخذنا تبويب

Example 1: $\mathbb{Z}_6$ is regular ring?

Sol: $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$
 $a = a \cdot b \cdot a$
 $0 = 0 \cdot 0 \cdot 0 = 0$
 $1 = 1 \cdot 1 \cdot 1 = 1$
 $2 = 2 \cdot 2 \cdot 2 = 8 - 6 = 2$
 $\vdots$
 $5 = 5 \cdot 5 \cdot 5 = 125 - 6 - 6 - \dots = 5$

* كل عناصر $\mathbb{Z}_6$ يجب ان تساوي قوتهم (الناتج ضربهم)

Example 2: $\mathbb{Z}_8$ is regular ring?

Sol: $\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$
 $a = a \cdot b \cdot a$
 $0 = 0 \cdot 0 \cdot 0 = 0$
 $1 = 1 \cdot 1 \cdot 1 = 1$
 $2 = 2 \cdot 2 \cdot 2 = 8 = 0 \neq 2 \neq a$

* تختبر كل عناصر

$\therefore \mathbb{Z}_8$ is not regular.

Example 1: $\mathbb{Z}_{12}$ is regular ring?

Soln $\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

$$a = a \cdot b \cdot a$$

$$0 = 0 \cdot 0 \cdot 0$$

$$1 = 1 \cdot 1 \cdot 1$$

$$2 = 2 \cdot 2 \cdot 2 = 8 \neq 2 \neq a$$

$\therefore \mathbb{Z}_{12}$ is not regular.

Example 2: $\mathbb{Z}_{18}$ is regular ring?

Soln $\mathbb{Z}_{18} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17\}$

$$a = a \cdot b \cdot a$$

$$0 = 0 \cdot 0 \cdot 0$$

$$1 = 1 \cdot 1 \cdot 1$$

$$2 = 2 \cdot 5 \cdot 2 = 20 - 18 = 2$$

$$3 = 3 \cdot 3 \cdot 3 = 27 - 18 = 9 \neq a$$

$\therefore \mathbb{Z}_{18}$ is not regular

* نأخذ قيمة a وقلنا على a

جرب كل الرقم a

الآن ان يظهر الرقم a المشابه

Example 3: $\mathbb{Z}_5$ is regular ring?

Soln $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$

$$a = a \cdot b \cdot a$$

$$0 = 0 \cdot 0 \cdot 0$$

$$1 = 1 \cdot 1 \cdot 1$$

$$2 = 2 \cdot 3 \cdot 2 = 12 - 5 - 5 = 2$$

$$3 = 3 \cdot 2 \cdot 3 = 18 - 5 - 5 - 5 = 3$$

$$4 = 4 \cdot 4 \cdot 4 = 64 - 5 - 5 - 5 - 5 - 5 = 4$$

SHAFAGH

$\therefore \mathbb{Z}_5$ is regular

Lemma: if R is regular ring then R/I is regular ring.

proof: let $a+I \in R/I$
Since $a \in R$, $a = a \cdot b \cdot a$

$$\begin{aligned} a+I &= a \cdot b \cdot a + I \\ &= (a+I)(b+I)(a+I) \\ \therefore R/I &\text{ is regular ring} \end{aligned}$$

Theorem: R ring Ideal $I = (n)$ R/I
 $R/(n) \cong \mathbb{Z}_n$

برهان: $R/(n) \cong \mathbb{Z}_n$ فكل مثالي على $\mathbb{Z}$

Q/give example prove that R is not regular but R/I is regular?

Sol: not regular. $R = \mathbb{Z}$ فكل مثالي على $\mathbb{Z}$ R/I is regular.

Example: The ring $(\mathbb{Z}, +, \cdot)$, $I = (3) = \{0, 3, 6, \dots\}$

Find $\mathbb{Z}/(3)$ is regular ring?
Sol: $\mathbb{Z}/I = \mathbb{Z}/(3) \cong \mathbb{Z}_3$

prove that $(\mathbb{Z}/(3))$ is regular

$$\mathbb{Z}/(3) = \{0+(3), 1+(3), 2+(3)\} \cong \mathbb{Z}_3$$

$$\mathbb{Z}_3 = \{0, 1, 2\}$$

$$a = a \cdot b \cdot a$$

$$0 = 0 \cdot 0 \cdot 0 \iff 0+(3) = (0+(3)) \cdot (0+(3)) \cdot (0+(3))$$

$$1 = 1 \cdot 1 \cdot 1 \iff 1+(3) = (1+(3)) \cdot (1+(3)) \cdot (1+(3))$$

$$2 = 2 \cdot 2 \cdot 2 \iff 2+(3) = (2+(3)) \cdot (2+(3)) \cdot (2+(3))$$

قابلية الاختزال reducible

Def: let R be a ring and I be any ideal of R . then I is said to be reducible if $\exists K$ and J are ideal of R

S.t:..
 1# $I \subset J$; $I \neq J$
 2# $I \subset K$; $I \neq J$ then $I = J \cap K$

Example: in $(\mathbb{Z}, +, \cdot)$ $I_1 = (3)$, $I_2 = (2)$

Find () is reducible?

Solu

$$\begin{aligned} (3) &= \{0, 3, 6, 9, \dots\} \\ (2) &= \{0, 2, 4, 6, \dots\} \\ (6) &\subset (3) \\ (6) &\subset (2) \\ (3) \cap (2) &= (6) \text{ reducible.} \end{aligned}$$

Example: in $(\mathbb{Z}, +, \cdot)$ $I_1 = (4)$, $I_2 = (2)$

Solu

$$\begin{aligned} (8) &\subset (2) \\ (8) &\subset (4) \\ (2) \cap (4) &= (8) \text{ reducible.} \end{aligned}$$

Lemma: Every ideal of regular ring is regular.

Proof: let I be an ideal of R , and $x \in I$ Since R is regular,

$$\exists y \in R \text{ s.t. } x = x \cdot y \cdot x$$

$$\text{if } z = yxy \in I$$

$$\begin{aligned} xzx &= xyxyx \\ &= (xyx)yx \\ &= xyx = x \end{aligned}$$

$$z = yxy$$

not regular

$\therefore I$ is regular.

essential

Def: we say that the ideal I is essential if $I \cap J \neq \{0\}$ for all J of R , if $I \cap J = 0$ then $J = (0)$.

Example: in $(\mathbb{Z}_6, +, \cdot)$ is ring is essential?

Sol:
 $I_1 = (2) = \{0, 2, 4\}$
 $I_2 = (3) = \{0, 3\}$

$$I_1 \cap I_2 = (2) \cap (3) = 0$$

$\therefore I_1, I_2$ is not essential

Example: in $(\mathbb{Z}_8, +, \cdot)$ is ring is essential?

Sol:
 $I_1 = (2) = \{0, 2, 4, 6\}$
 $I_2 = (4) = \{0, 4\}$

$$I_1 \cap I_2 = (2) \cap (4) = \{0, 4\} = (4) \neq 0$$

$\therefore I_1, I_2$ is essential

Example: $(Z_{12}, +, \cdot)$ is ring — is essential

Sol

$$I_1 = (2) = \{0, 2, 4, 6, 8, 10\}$$

$$I_2 = (3) = \{0, 3, 6, 9\}$$

$$I_3 = (4) = \{0, 4, 8\}$$

$$I_4 = (6) = \{0, 6\}$$

Find $(2) \cap (3)$?

$(2) \cap (4)$?

$(2) \cap (6)$?

$(3) \cap (4)$?

$$(2) \cap (3) = (6)$$

$$(2) \cap (4) = (8)$$

$$(2) \cap (6) = (6)$$

$$(3) \cap (4) = (0)$$

$\therefore (2)$ is essential.

$(3), (4), (6)$ not essential.

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Uniform..

Def: A ring R is said to be uniform if for any ideal of R is essential.

Example: $(\mathbb{Z}_8, +, \cdot)$ is ring -- is uniform?

Soln $I_1 = (2) = \{0, 2, 4, 6\}$

$I_2 = (4) = \{0, 4\}$

$I_1 \cap I_2 = (2) \cap (4) = (4) \therefore \mathbb{Z}_8 \text{ is essential}$

$\therefore \mathbb{Z}_8 \text{ is uniform.}$

لأننا كانت الناتج للتقاطع $\neq 0$

essential إذا

uniform. وإذا كانت essential إذا

Example: $(\mathbb{Z}_6, +, \cdot)$ is ring -- is uniform?

Soln $I_1 = (2) = \{0, 2, 4\}$

$I_2 = (3) = \{0, 3\}$

$I_1 \cap I_2 = (2) \cap (3) = 0$

$\therefore \mathbb{Z}_6 \text{ is not essential}$

$\therefore \mathbb{Z}_6 \text{ is not uniform}$